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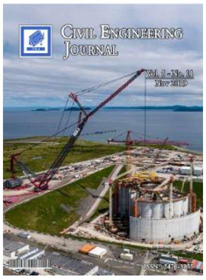

Footing Soil Pressure from Biaxial Loading

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Abstract

A symmetrical isolated rectangular footing with centered biaxial overturning develops soil pressure that shifts to counter balance the loads. The highest soil pressure is at a corner. The objective of this paper is to extend the uniaxial soil pressure solution to include biaxial loads and to document a simple and understandable way to directly calculate the shape of the soil pressure distribution. Another objective is to make the solution suitable for automation. In uniaxial overturning there are two transition shapes, trapezoidal and triangular. In biaxial overturning there are three transition shapes and they form 4, 5 & 6 sided polyhedrons. This analysis calculates those volumes and compares them to the design vertical load to determine the characteristic shape of the soil pressure distribution. The calculation then proceeds to converge on the precise shape and calculate its centroid and moment capacity. Assemblies of tetrahedrons are used to model all of the soil pressure shapes. The advantage of this methodology is that matrix algebra can be used to organize the calculations and make them computationally efficient. The assumed soil pressure and footing dimensions can be adjusted until the calculated moment capacity matches the overturning moment.

Keywords: Footing; Soil Pressure; Biaxial; Tetrahedron; Determinant.

1. Introduction

A symmetrical isolated rectangular footing with centered biaxial overturning loads develop soil pressure that shifts to counter balance the loads. The highest soil pressure is at a corner.

Figure 1 shows a footing with biaxial loading. All the loads are resolved to the center of the footing and they are perpendicular to one another. For soil pressure calculations all of the loads are projected to the foundation soil contact plane.

The input variables are as follows;

- V_f Footing vertical load
- W Footing Width
- L Footing Length
- MPL Moment parallel
- MPR Moment perpendicular

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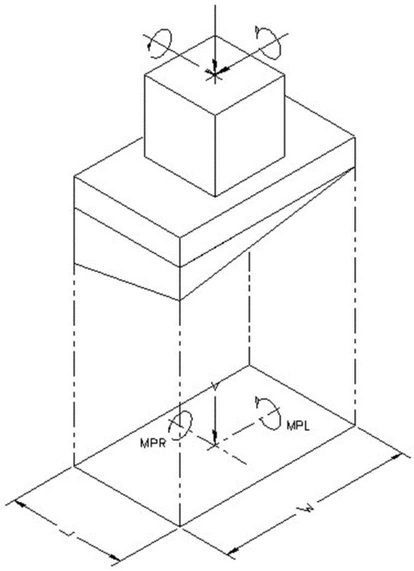

Figure 1. Loads at foundation contact

In plan, Peck [1] presents the axis of zero soil pressure as a line based on the pressure from eccentricity in each direction (Figure 2). Equation 1 is used to compute the fictitious pressures at the corners as if tension across the soil footing contact plane could exist. The moments create highest soil pressure at the corner labeled 1.

$$q = \frac{V}{A} \pm \frac{MPR * cx}{Iy} \pm \frac{MPL * cy}{Ix} \tag{1}$$

The axis of zero soil pressure is then located on the basis of pressures calculated from Equation 1. Including the moments of inertia accounts for the tendency of the footing to roll toward the weak axis. This is most evident for a footing with significant differences in the dimensions of L and W. For a square footing with length and width the same, the axis of zero soil pressure is aligned perpendicular to the moment vector.

$$\emptyset = \text{atan}\left(\frac{\frac{MPR * cx}{Iy}}{\frac{MPL * cy}{Iy}}\right) \tag{2}$$

$$M = \sqrt{MPR^2 + MPL^2} \tag{3}$$

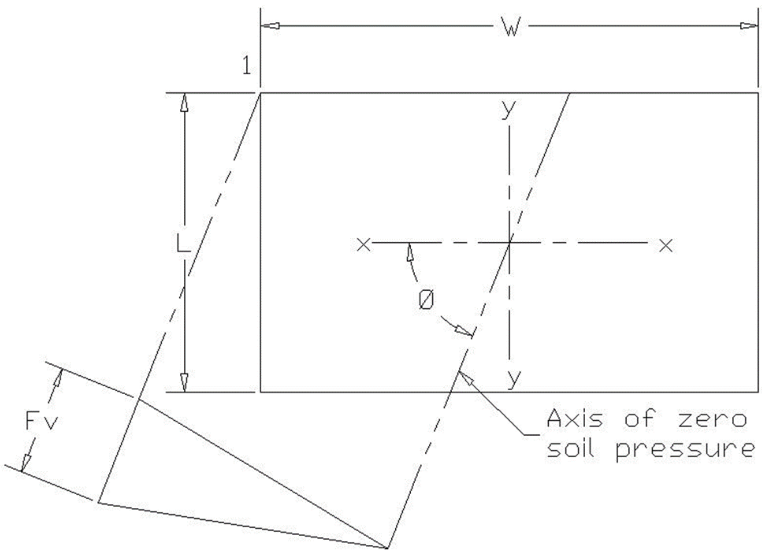

Figure 2. Axis of zero soil pressure

In isometric view (Figure 3) the characteristic shape of the soil pressure is revealed (In this view the shape is between the first and second transitions). The shape grows in the direction of +V or shrinks in the direction of –V until the volume of the pressure distribution matches the vertical load. The base of the soil pressure distribution slopes as a planer surface between the maximum soil pressure and the axis of zero soil pressure. The soil pressure volume grows or shrinks as the axis of zero soil pressure moves parallel to itself.

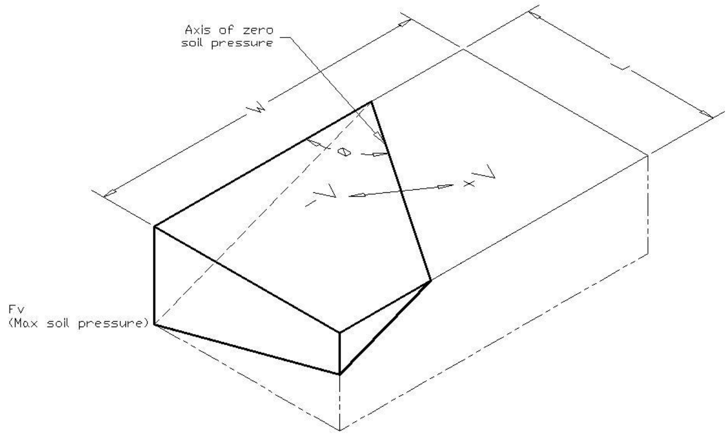

Figure 3. Soil pressure shape

1.1. Uniaxial Overturning Soil Pressure

When a footing is subject to vertical load without overturning, the pressure distribution is uniform across the base of the footing. When overturning is applied the soil pressure shifts to counterbalance the moment. The volume shifts away from uniform and through two transition volumes, first trapezoidal and then triangular (Figure 4). Through all transitions the volume of the pressure distribution is still equal to the vertical load. In each case the vertical load multiplied by the eccentricity equals the overturning moment [2].

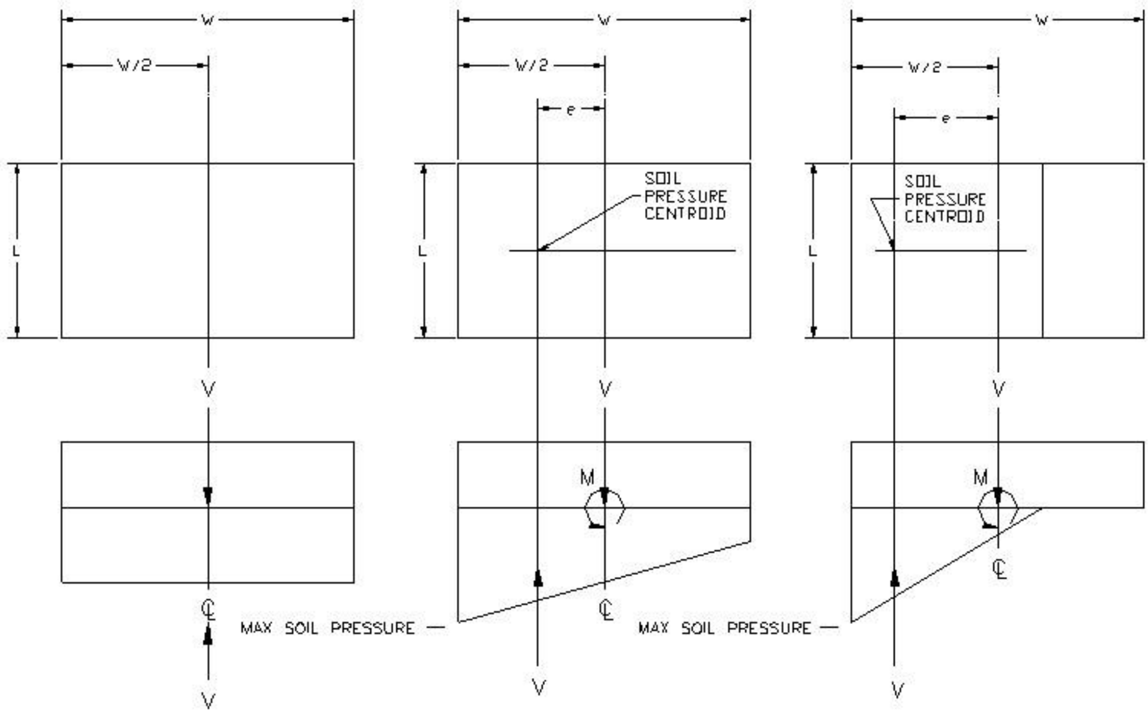

Figure 4. Uniaxial soil pressure

1.2. Biaxial Overturning

The same principal applies to biaxial overturning. Instead of two transition shapes there are three. The orientation of the axis of zero soil pressure is calculated (Equation 2). The leading edge of the soil pressure distribution or the axis of zero soil pressure is the intersection between the footing/soil contact plane and the base of the pressure distribution plane. As the axis of zero soil pressure moves parallel to itself, the soil pressure volume grows from enclosing small volumes to larger volumes. As is crosses through the corners it forms the transition volumes. It crosses three corners, thus there are three transition volumes.

When $L < W * \tan(\emptyset)$ the axis of zero soil pressure contacts the corner at L before W (Figure 5). When the axis of zero soil pressure contacts W before L then the solution follows a parallel but separate path or a simple axis rotation can be made to use this solution.

For each foundation and assumed maximum soil bearing pressure Fv, there are three transition volumes dependent only on V_f, W, L and the axis of zero soil pressure ϕ. The three transitions for an arbitrary angle are shown in Figure 5. When their volumes are calculated and compared to the footing vertical load the characteristic shape for the soil pressure distribution is determined. The characteristic shape remains the same between transitions. Up to the first transition, the shape of the volume is four sided. Between the first and second transitions, its five sided. Between the second and third transitions, its six sided. And when greater than the third transition it's still six sided.

The range of volumes through all transitions is from uniform soil pressure and no overturning on the high end, to small tetrahedrons with highest possible eccentricities on the low end. When the footing vertical load is compared to the range of possible volumes, the characteristic soil pressure shape for the footing is found. Once identified, iteration is used to converge on the precise shape that matches the footing vertical load. The centroid of the shape is then calculated to determine the moment capacity. The footing and the loads are a free body and the calculation simply balances the forces and moments [7].

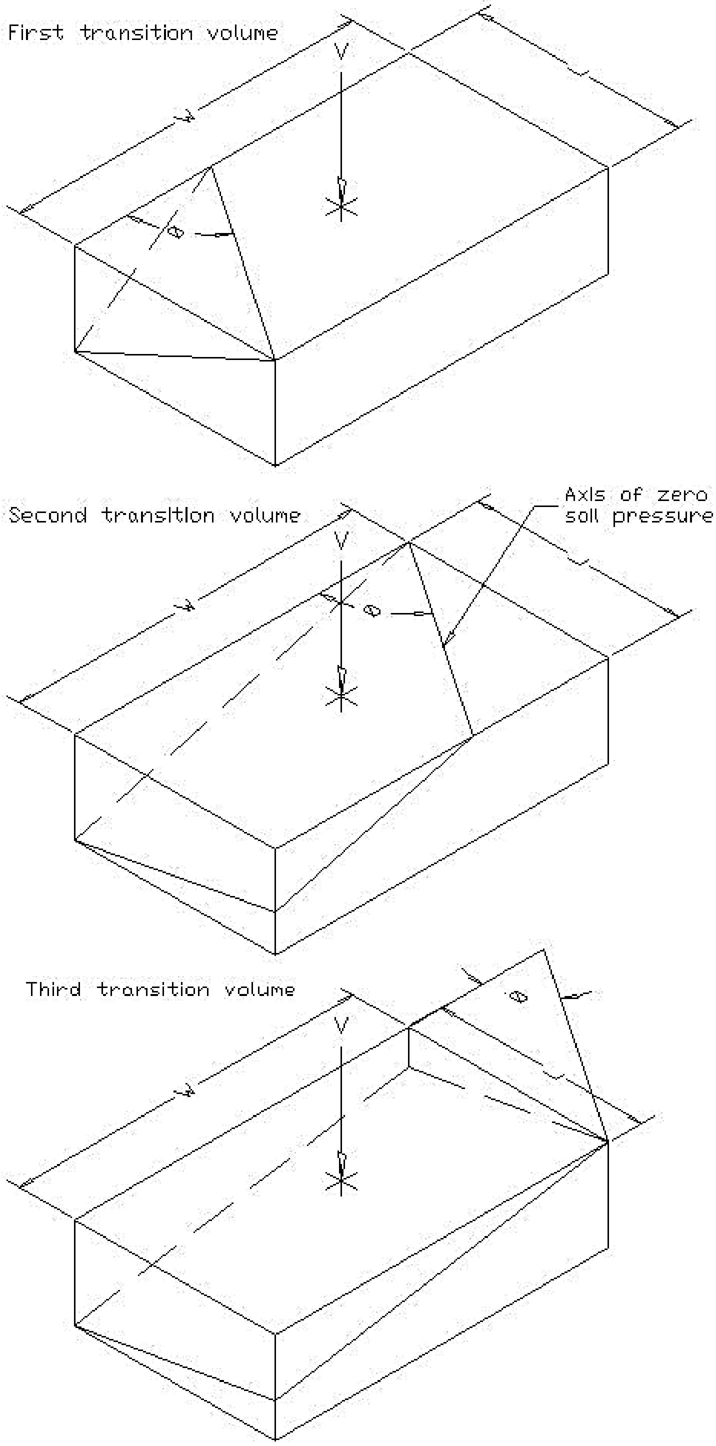

Figure 5. Three transition volumes

1.3. Maximum Soil Pressure

The maximum soil pressure is the peak pressure that the foundation exerts on the ground under the foundation in units of force per length squared. Limits on this value are dictated by site conditions. For this paper it is assumed that limits on maximum soil pressure are provided separately. For analysis purposes, with known footing dimensions and loads, the maximum soil pressure is adjusted until the calculated moment capacity matches the overturning. For design purposes, site limitations or allowable soil bearing pressure are initially assumed to size the foundation. Once the foundation is sized to meet design limitations and requirements, further refinement of the maximum soil pressure can be made to determine the smallest value that balances the overturning load. A footing that meets allowable soil pressure limits may have actual soil pressure that is lower.

A footing subject to significant biaxial overturning must be sized to allow soil pressure to build at the corner. If the allowable soil bearing pressure is significantly higher than the unloaded condition (V/A) higher overturning can be accommodated. The footing must be big enough and strong enough to take advantage of the difference between the unloaded condition and the highest allowable pressure at the corner with overturning. The most economical solution is where the soil pressure reaches the allowable and vertical load eccentricity balances the overturning. The economical solution takes advantage of the allowable soil bearing pressure by reducing the footing size while achieving other stability goals and forcing the corner bearing pressure toward the allowable. If reserve soil bearing exists, the solution here will move toward the first transition volume. If the footing vertical load is small relative to the overturning moments the solution will also move toward the first transition volume.

1.4. Calculation Method

For this method of calculating soil pressure the loads and footing dimensions are known and the maximum soil pressure assumed. The calculation then proceeds to calculate the three transition volumes associated with those values. The design vertical load is then compared to the calculated transition volumes.

Here, the calculation branches to one of four paths. The first path is followed when the footing vertical load is less than the first transition volume. The second path is followed when the footing vertical load is between the first and second transition volumes. The third path is followed when the footing vertical load is between the second and third transition volumes and the fourth path is used when the footing vertical load is greater than the third transition volume. Although the solution is broken into four paths, the soil pressure shape transition is continuous from uniform soil pressure to a corner tetrahedron. The footing soil contact plane rotates about the highest loaded corner parallel to the axis of zero pressure. It starts horizontal and it ends as it approaches vertical. The transition volumes are the places it passes the other foundation corners. The soil pressure shape is the intersection of those two planes and vertical planes from the sides of the footing [4]. It is important to recognize the planer structure in order to reduce the number of unknown variables.

For all four calculation paths, all of the corner points of the pressure distribution are written in terms of a single variable that is adjusted until the equation for the volume converges on the design vertical load. With just a single variable, there is a direct solution. It just produces a large complicated equation for both the volume and the center of gravity. No attempt is made here to solve for the single variable, rather this paper relies on the iterative solution. This makes it easier to control the precision, particularly when supported by automation. Consider using different precisions for z axis variables to adjust for units. Following convergence on the footing vertical load, the center of gravity and the moment capacity are calculated.

Several analytical and graphical methods for solving this problem are available [1, 5-8] but the introduction of tetrahedrons and matrices simplifies the solution. There are certainly other numerical and higher level math solutions. This solution is methodical and direct.

The results of the calculation is a moment capacity for the given variables and the assumed soil bearing pressure. If the moment capacity is too low, the maximum soil pressure is increased or the dimensions of the footing are increased. If the moment capacity is too high, the maximum soil pressure is decreased or the dimensions are decreased. All solutions with excess moment capacity are viable, but not the most economic. The optimal solution is found when the highest allowable soil pressure is used and the moment capacity balances the overturning moment. To lower a footings soil pressure add size. Adding footing size in the direction of a line perpendicular to the axis of zero soil pressure is most efficient in counterbalancing overturning. Peck [1] provided a caution “However, a great degree of refinement cannot be justified in view of the uncertainties associated with the various assumptions that must be made in solving any problem of this type”.

A flowchart of this design process is shown in Figure 6.

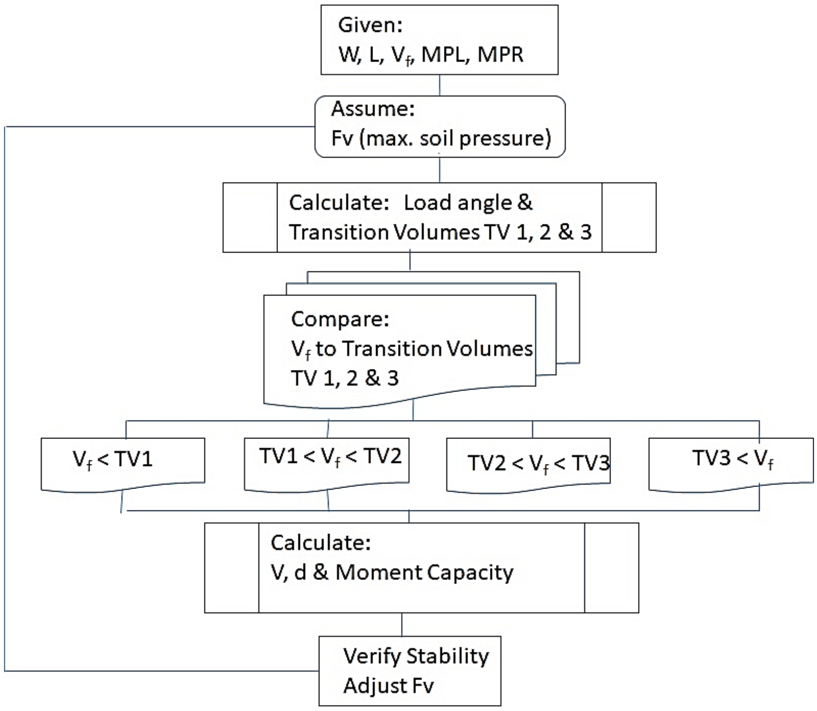

Figure 6. Calculation flowchart

2. Transition volumes

2.1. Transition Volume 1

The first transition volume (TV1) forms a tetrahedron with coordinates shown in Figure 7. The determinant of the tetrahedron corner matrix yields the volume [9, 10]. The volume of all tetrahedrons is positive so absolute values of each determinant is used. The dimension W is aligned with the X axis, L with the Y axis and Fv with the Z axis.

Corner coordinates			
Point Number	X (ft)	Y (ft)	Z (lb/sq-ft)
1	0	0	0
2	X2	0	0
3	0	L	0
4	0	0	Fv

$$\mathrm{X2} = \mathrm{L}/\tan\phi \quad (4)$$

$$\mathrm{V} = \frac{1}{6} * \begin{bmatrix} 0 & 0 & 0 & 1 \\ X2 & 0 & 0 & 1 \\ 0 & L & 0 & 1 \\ 0 & 0 & Fv & 1 \end{bmatrix} \quad (5)$$

$$\mathrm{V} = \frac{1}{6} * (Fv * L * X2) \quad (6)$$

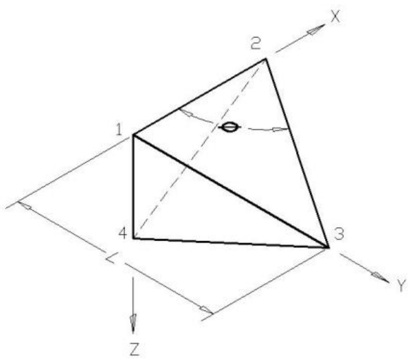

Figure 7. First transition volume

2.2. Transition Volume 2

As the soil pressure volume grows past the first transition to the second, the projections of lines 2-3, 1-4 and 6-5 and the planes associated with them intersect at point A (Figure 8) [4]. This provides a visual model for the solution of the variables. The variables X3 and Z5 are written in terms of X2 and known values (Equations 5 & 6) [11].

The second transition volume (TV2) is comprised of three independent tetrahedrons with corner coordinates shown in Figure 8. For each tetrahedron, a corner matrix is formed for the calculations. The volume is based on the sum of the determinants of the matrices. Other tetrahedron assemblies may be used to build the same volume, but the tetrahedron assemblies must be independent and fully encompasses, without gaps, the soil pressure volume.

Corner coordinates			
Point Number	X (ft)	Y (ft)	Z (lb/sq-ft)
1	0	0	0
2	W	0	0
3	X3	L	0
4	X2	0	0
5	0	L	Z5
6	0	0	Fv

$$\mathrm{X3} = \mathrm{W} - \frac{\mathrm{L}}{\tan(\phi)} \tag{7}$$

$$\mathrm{Z5} = \mathrm{Fv} * \frac{\mathrm{W} * \tan(\phi) - \mathrm{L}}{\mathrm{W} * \tan(\phi)} \tag{8}$$

Figure 8 includes dotted lines delineating the tetrahedrons. The tetrahedron corners are listed here. The first three corners can be viewed as a triangle. The fourth corner can be viewed as the apex or common point of the triangle vertices. But any three corners can be viewed as the triangle, with the fourth corner being the apex.

- 1, 6, 2, 3
- 1, 3, 4, 6
- 4, 5, 3, 6

$$\mathrm{V} = \frac{1}{6} * \left[\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & Fv & 1 \\ W & 0 & 0 & 1 \\ X3 & L & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ X3 & L & 0 & 1 \\ 0 & L & 0 & 1 \\ 0 & 0 & Fv & 1 \end{bmatrix} + \begin{bmatrix} 0 & L & 0 & 1 \\ 0 & L & Z5 & 1 \\ X3 & L & 0 & 1 \\ 0 & 0 & Fv & 1 \end{bmatrix} \right]$$

The volume equations separate each determinant by a plus (+) sign.

$$\mathrm{V} = \frac{1}{6} * (Fv * L * W + Fv * L * X3 + L * X3 * Z5)$$

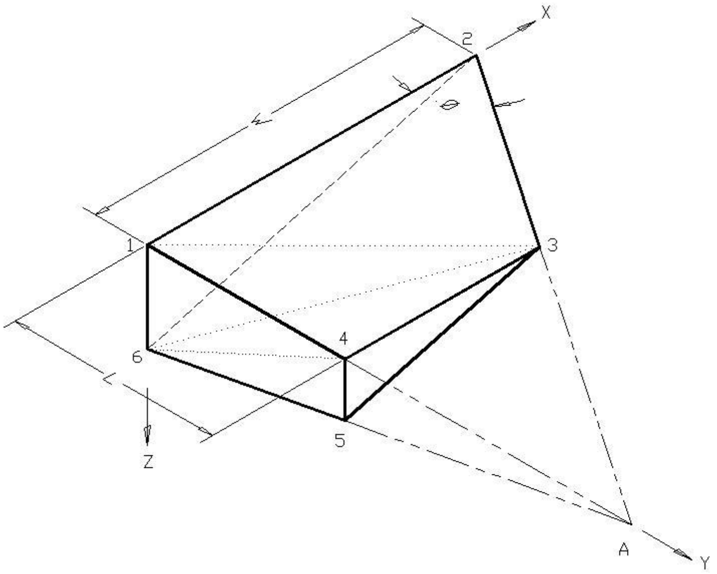

Figure 8. Second Transition Volume

2.3. Transition Volume 3

As the soil pressure volume grows past the second transition to the third, the projections of lines 1-4 and 6-5 and the planes associated with them intersect at point A and the projections of lines 1-2 and 6-7 and the planes associated with them intersect at point B. This provides a visual model for the solution of the variables (Figure 9). The variables Z5 and Z7 are written in terms of W, L and Fv from similar triangles.

The third transition volume (TV3) is comprised of four independent tetrahedrons with coordinates shown in Figure 9. The sum of the determinants of the four tetrahedron corner matrices yields the volume which from symmetry is always exactly half of the full volume ($Fv \times W \times L$).

Corner coordinates			
Point Number	X (ft)	Y (ft)	Z (lb/sq-ft)
1	0	0	0
2	W	0	0
3	W	L	0
4	0	L	0
5	0	L	Z5
6	0	0	Fv
7	W	0	Z7

$$\mathrm{Z5} = \mathrm{Fv} * \frac{\mathrm{W} * \tan(\phi)}{L + W * \tan(\phi)} \tag{11}$$

$$\mathrm{Z7} = \frac{\mathrm{Fv} * \mathrm{L}/\tan(\phi)}{\mathrm{W} + \mathrm{L}/\tan(\phi)} \tag{12}$$

Tetrahedron corners.

- 1, 2, 3, 6
- 1, 6, 3, 4
- 2, 3, 7, 6
- 4, 5, 6, 3

The Soil pressure volume is based on the sum of the determinants of the four tetrahedron corner matrices.

$$\mathrm{V} = \frac{1}{6} * \left[\begin{bmatrix} 0 & 0 & 0 & 1 \\ W & 0 & 0 & 1 \\ W & L & 0 & 1 \\ 0 & 0 & Fv & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & Fv & 1 \\ W & L & 0 & 1 \\ 0 & L & 0 & 1 \end{bmatrix} + \begin{bmatrix} W & 0 & 0 & 1 \\ W & L & 0 & 1 \\ W & 0 & Z7 & 1 \\ 0 & 0 & Fv & 1 \end{bmatrix} + \begin{bmatrix} 0 & L & 0 & 1 \\ 0 & L & Z5 & 1 \\ 0 & 0 & Fv & 1 \\ W & L & 0 & 1 \end{bmatrix} \right] \tag{13}$$

$$\mathrm{V} = \frac{1}{6} * (W * L * Fv + W * Fv * L + W * L * Z7 + W * L * Z5) \tag{14}$$

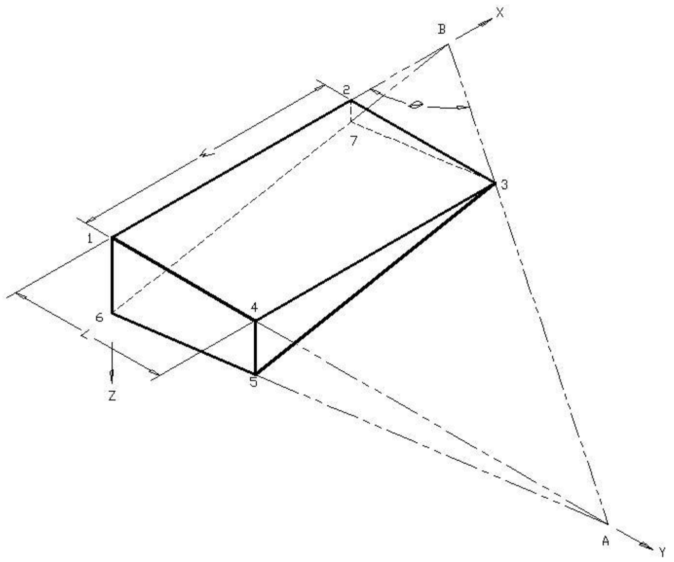

Figure 9. Third Transition Volume

3. Soil Pressure Solutions

This is where a decision is made as to which calculation path to follow. The three transition volumes have been calculated. The footing vertical load is known. One of four calculation paths is chosen that is representative of the relationship between the footing vertical load and the transition volumes.

3.1. 0 < Footing Vertical Load < First Transition Volume

As Y3 transitions from zero to L, the volume grows from zero to the first transition volume (Figure 10).

Corner coordinates			
Point Number	X (ft)	Y (ft)	Z (lb/sq-ft)
1	0	0	0
2	X2	0	0
3	0	Y3	0
4	0	0	Fv

$$\mathrm{X2} = Y3/\tan(\emptyset) \tag{15}$$

The volume of the soil pressure distribution in terms of Y3 is based on the determinant of the tetrahedron corner matrix.

$$\mathrm{V} = \frac{1}{6} * \begin{bmatrix} 0 & 0 & 0 & 1 \\ X2 & 0 & 0 & 1 \\ 0 & Y3 & 0 & 1 \\ 0 & 0 & Fv & 1 \end{bmatrix} \tag{16}$$

$$\mathrm{V} = \frac{1}{6} * (Fv * X2 * Y3) \tag{17}$$

The center of gravity of the volume in the plan view plane are shown in Equation 18 & 19 [9].

$$\mathrm{Xcg} = \frac{1}{4}\sum_{i=1}^{4} x_i \tag{18}$$

$$\mathrm{Ycg} = \frac{1}{4}\sum_{i=1}^{4} y_i \tag{19}$$

The x_i and y_i are taken directly from the matrix columns x and y.

$$\mathrm{Xcg} = \frac{X2}{4} \tag{20}$$

$$\mathrm{Xcg} = \frac{\mathrm{Y3}}{4} \tag{21}$$

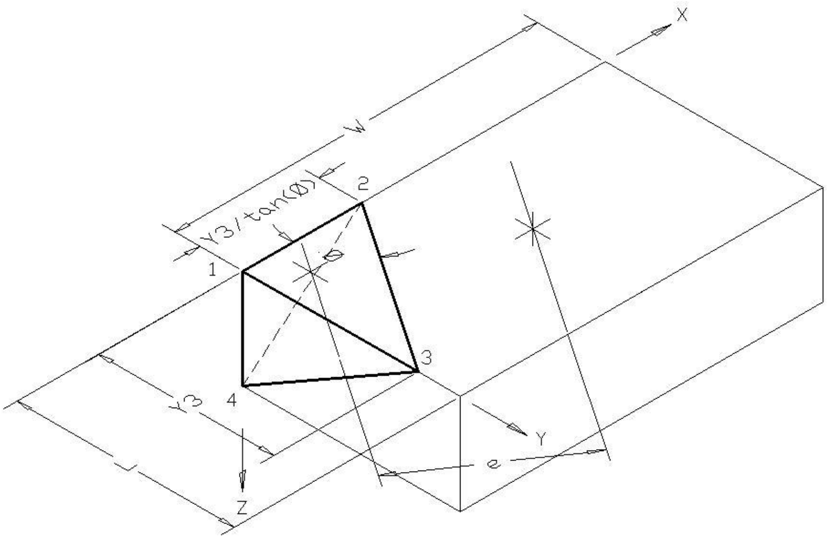

Figure 10. Footing vertical load (V_f) < first transition volume

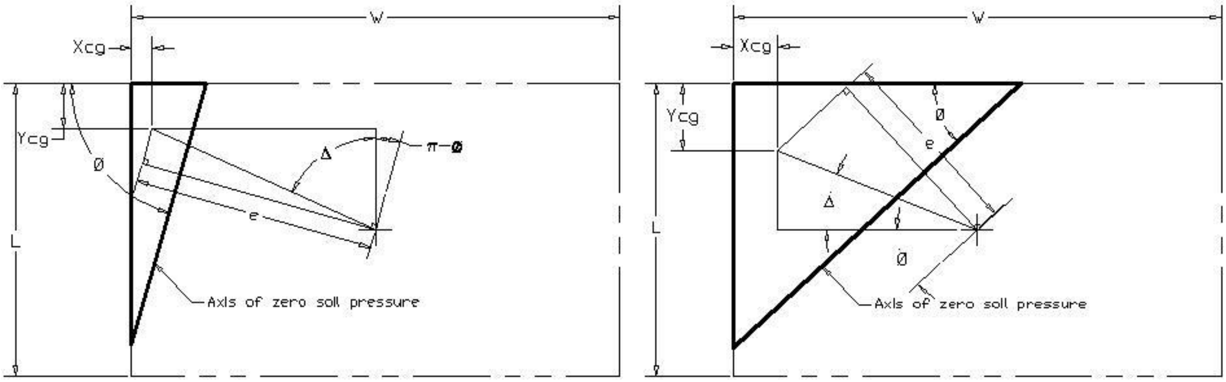

Figure 11. Eccentricity

3.2. Eccentricity (e)

The footing eccentricity is the perpendicular distance between the center of gravity of the soil pressure distribution and the center of the footing. The calculation for the eccentricity depends on the relative position between a line drawn from the center of the footing to the centroid of the pressure shape and another line drawn from the center of the footing perpendicular to a line parallel to the axis of zero soil pressure passing through the centroid. Both possibilities are shown in Figure 11. Angle ϕ has a range of $\pi/2$ to $Atan(L/W)$ which swings past the line from the center to the center of gravity. Equation 22 accounts for this and it and Equation 23 are applicable for all four solution paths.

$$\text{If } \phi > \pi - \operatorname{atan}\left(\frac{\frac{L}{2} - Ycg}{\frac{W}{2} - Xcg}\right) \text{ then}$$

$$e = \cos\left(\emptyset - \pi/2 - \operatorname{atan}\left(\frac{\frac{W}{2} - Xcg}{\frac{Y}{2} - Ycg}\right)\right) * \sqrt{\left(\frac{W}{2} - Xcg\right)^2 + \left(\frac{L}{2} - Xcg\right)^2} \tag{22}$$

Else

$$e = \cos\left(\pi/2 - \emptyset - \operatorname{atan}\left(\frac{\frac{L}{2} - Ycg}{\frac{W}{2} - Xcg}\right)\right) * \sqrt{\left(\frac{W}{2} - Xcg\right)^2 + \left(\frac{L}{2} - Xcg\right)^2}$$

Footing capacity

$$M = e * V \tag{23}$$

3.3. First Transition Volume < Footing Vertical Load < Second Transition Volume

As X2 grows from L/tan(ϕ) to W, the volume grows from the first transition to the second transition volume (Figure 12). As the x coordinate of point 3 grows, the projections of the line 2-3, 1-4 and 6-5 and the planes associated with them intersect at a point. The projected triangles form a visual basis for calculation of the variables X3 and Z5. They are calculated in terms of X2 (Equations 24 & 25). The volume of the soil pressure volume is based on the sum of the determinants of the three corner matrices (Equations 26 & 27). The eccentricity (e) is used to calculate the moment capacity.

Corner coordinates			
Point Number	X (ft)	Y (ft)	Z (lb/sq-ft)
1	0	0	0
2	X2	0	0
3	X3	L	0
4	0	L	0
5	0	L	Z5
6	0	0	Fv

Variables in terms of X2

$$X3 = X2 - \frac{L}{\tan(\emptyset)} \tag{24}$$

$$Z5 = Fv * \frac{X2 * \tan(\emptyset) - L}{X2 * \tan(\emptyset)} \tag{25}$$

Tetrahedron corners.

- 1, 6, 2, 3
- 1, 3, 4, 6
- 4, 5, 3, 6

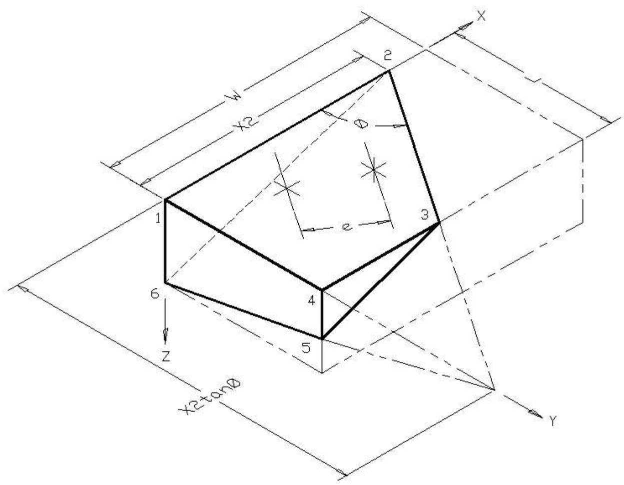

Figure 12. First transition volume < (V_f) < second transition volume

The volume of the soil pressure distribution in terms of X2 is based on the sum of the volume of the three tetrahedrons.

$$V = \frac{1}{6} * \left[\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & Fv & 1 \\ X2 & 0 & 0 & 1 \\ X3 & L & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ X3 & L & 0 & 1 \\ 0 & L & 0 & 1 \\ 0 & 0 & Fv & 1 \end{bmatrix} + \begin{bmatrix} 0 & L & 0 & 1 \\ 0 & L & Z5 & 1 \\ X3 & L & 0 & 1 \\ 0 & 0 & Fv & 1 \end{bmatrix} \right] \tag{26}$$

$$V = \frac{1}{6} * (X2 * L * Fv + Fv * L * X3 + L * X3 * Z5) \tag{27}$$

For soil pressure volumes comprised of multiple tetrahedrons, geometric decomposition is applied [12].

$$Xcg = \frac{\sum_1^n (V_i * Xcg_i)}{V} \tag{28}$$

The center of gravity of the volume is derived from the assembly of individual tetrahedrons.

$$Xcg = \frac{1}{V * 6} * \left(X2 * L * Fv * \frac{X2 + X3}{4} + Fv * L * X3 * \frac{X3}{4} + Fv * L * X3 * Fv * Z5 * \frac{X3}{4} \right) \tag{29}$$

$$Ycg = \frac{1}{V * 6} * \left(X2 * L * Fv * \frac{L}{4} + Fv * L * X3 * \frac{2 * L}{4} + L * X3 * Z5 * \frac{3 * L}{4} \right) \tag{30}$$

The footing eccentricity (e) is the perpendicular distance between the center of gravity of the soil pressure distribution and the center of the foundation. See equations 22 and 23.

3.4. Second Transition Volume < Footing vertical load < Third transition volume

As dimension Y3 grows from zero to L, the volume grows from the second transition to the third transition volume (Figure 13). As it grows, the projections of lines 1-5, 6-8 and 3-4 meet at point A. The projections of line 1-2, 6-7 and 3-4 meet at point B. The projected triangles form a visual basis for calculation of the variables. X4, Z7 and Z8 are calculated in terms of Y3 (Equations 31, 32 & 33).

Corner coordinates			
Point Number	X (ft)	Y (ft)	Z (lb/sq-ft)
1	0	0	0
2	W	0	0
3	W	Y3	0
4	X4	L	0
5	0	L	0
6	0	0	Fv
7	W	0	Z7
8	0	L	Z8

Variables in terms of Y3

$$\mathrm{X4} = \mathrm{W} - \left(\frac{\mathrm{L} - \mathrm{Y3}}{\tan(\emptyset)}\right) \tag{31}$$

$$\mathrm{Z7} = \frac{\mathrm{Fv} * \mathrm{Y3}}{\mathrm{W} * \tan(\emptyset) + Y3} \tag{32}$$

$$\mathrm{Z8} = \frac{\mathrm{Fv} * (\mathrm{W} * \tan(\emptyset) + \mathrm{Y3} - \mathrm{L})}{(\mathrm{Y3} + \mathrm{W} * \tan(\emptyset))} \tag{33}$$

Tetrahedron corners.

- 6, 2, 3, 7
- 1, 6, 3, 2
- 1, 6, 4, 3
- 6, 5, 4, 8
- 1, 6, 5, 4

The volume of the soil pressure distribution in terms of Y3 is based on the sum of the volume of the five tetrahedrons.

$$\mathrm{V} = \frac{1}{6} * \left[\begin{bmatrix} 0 & 0 & \mathrm{Fv} & 1 \\ W & 0 & 0 & 1 \\ W & Y3 & 0 & 1 \\ \mathrm{W} & 0 & Z7 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & Fv & 1 \\ W & Y3 & 0 & 1 \\ \mathrm{W} & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \mathrm{Fv} & 1 \\ X4 & \mathrm{L} & 0 & 1 \\ \mathrm{W} & \mathrm{Y3} & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \mathrm{Fv} & 1 \\ 0 & L & 0 & 1 \\ \mathrm{X4} & \mathrm{L} & 0 & 1 \\ 0 & L & \mathrm{Z8} & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & Fv & 1 \\ 0 & \mathrm{L} & 0 & 1 \\ X4 & L & 0 & 1 \end{bmatrix}\right] \tag{34}$$

$$\mathrm{V} = \frac{1}{6} * (W * Y3 * Z7 + \mathrm{W} * \mathrm{Y3} * Fv + Fv * (L * W - X4 * Y3) + L * X4 * Z8 + Fv * L * X4) \tag{35}$$

The center of gravity of the volume is:

$$\mathrm{Xcg} = \frac{1}{\mathrm{V} * 6} * \left(W * Y3 * Z7 * \frac{3 * W}{4} + \mathrm{W} * \mathrm{Y3} * Fv * \frac{2 * W}{4} + Fv * (L * W - X4 * Y3) * \frac{X4 + W}{4} + L * X4 * Z8 * \frac{X4}{4} + Fv * L * X4 * \frac{X4}{4}\right) \tag{36}$$

$$\mathrm{Ycg} = \frac{1}{\mathrm{V} * 6} * \left(W * Y3 * Z7 * \frac{Y3}{4} + \mathrm{W} * \mathrm{Y3} * Fv * \frac{Y3}{4} + Fv * (L * W - X4 * Y3) * \frac{L + Y3}{4} + L * X4 * Z8 * \frac{L + L + L}{4} + Fv * L * X4 * \frac{L + L}{4}\right) \tag{37}$$

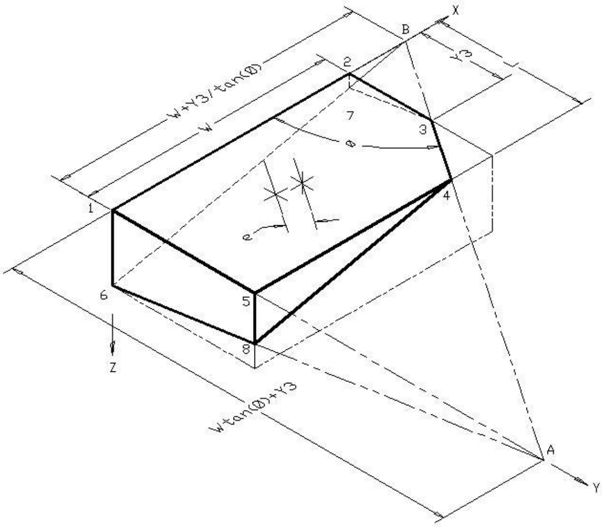

Figure 13. Second transition volume < (V_f) < third transition volume

The footing eccentricity (e) is the perpendicular distance between the center of gravity of the soil pressure distribution and the center of the foundation. See Equations 22 and 23.

3.5. Third Transition Volume < Footing Vertical Load

As coordinate Z5 grows from zero to Fv, the volume grows from the third transition to full volume (Figure 14). In this range there is compression on all of the foundation contact and there is a parallel separate solution of V/A±Vec/I. As the volume grows greater than the third transition volume the axis of zero soil pressure leaves the footing footprint. As this happens the projections of lines 1-2 and 6-7 meet at point B. The projections of line 1-4 and 6-8 meet at point A. The projected triangles form a visual basis for calculation of the variables Z7 and Z8. They are calculated in terms of Z5 (Equations 44 & 45). Before Z7 and Z8 can be calculated a set of lines and angles are developed in a section through the footing corners and on a plan view. The goal is to develop the projections of the foundation lines (a and c) to the Axis of zero soil pressure (Figures 15 and 16).

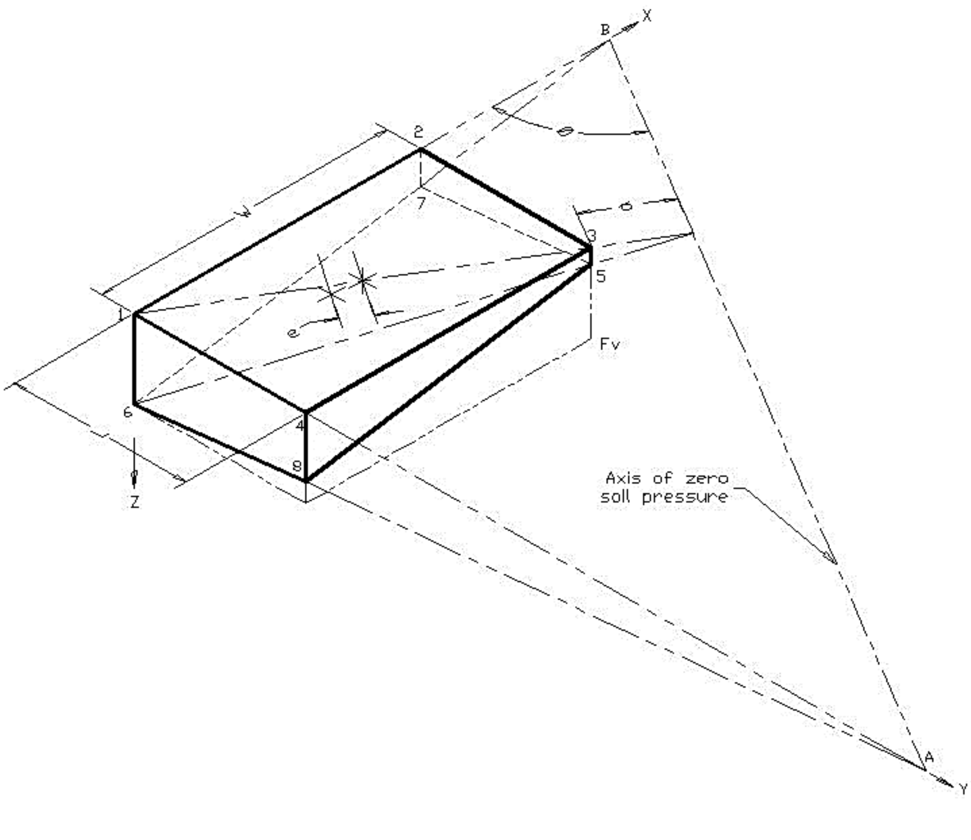

Figure 14. Isometric view Vf > third transition volume

Corner Coordinates			
Point Number	X (ft)	Y (ft)	Z (lb/sq-ft)
1	0	0	0
2	W	0	0
3	W	L	0
4	0	L	0
5	W	L	Z5
6	0	0	Fv
7	W	0	Z7
8	0	L	Z8

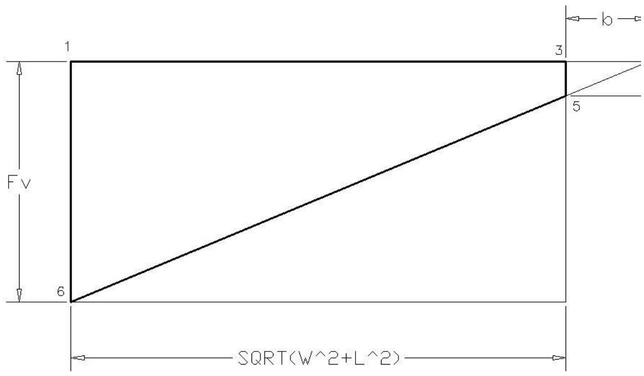

Figure 15. Section view V_f > third transition volume

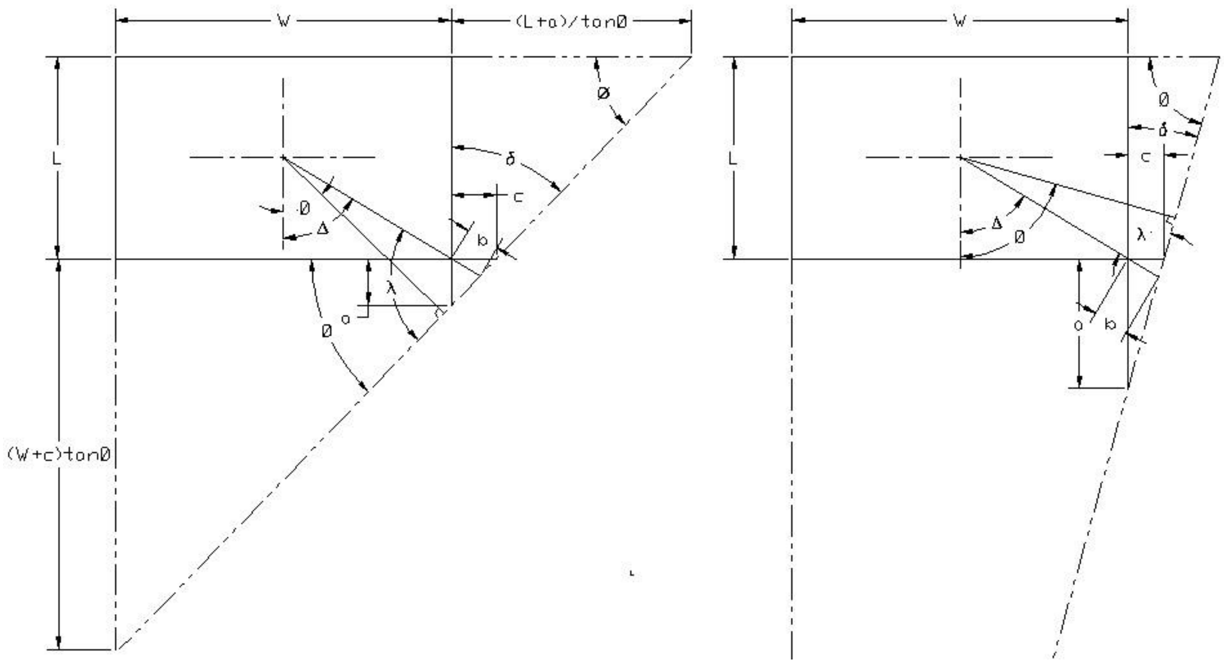

Figure 16. Plan view V_f > third transition volume

$$b = \frac{Z5 * \sqrt{W^2 + L\hat{}2}}{Fv - Z5} \tag{38}$$

$$\Delta = \mathrm{atan}\left(\frac{W}{L}\right) \tag{39}$$

$$\delta = \frac{\pi}{2} - \phi \tag{40}$$

In plan view, when a line is drawn from the center of the footing through the footing corner and intersecting the line of zero soil pressure, it forms angle Δ. The calculations for Z7 and Z8 depend on the relative positions between ϕ and Δ. Depending on the footing shape and the axis of zero soil pressure, ϕ can land on either side of Δ. Both possibilities are shown in Figure 16. Equations 41, 42 and 43 account for that.

$$\text{If } \phi < \Delta\text{: } \lambda = \frac{\pi}{2} - (\Delta - \phi) \text{ Else } \lambda = \frac{\pi}{2} - (\phi - \Delta) \tag{41}$$

$$\text{If } \phi < \Delta\text{: } a = \frac{b * \sin(\delta)}{\sin(\lambda)} \text{ Else } a = c * \tan(\emptyset) \tag{42}$$

$$\text{If } \phi > \Delta\text{: } c = a/\tan(\phi) \text{ Else } c = \frac{b * \sin(\lambda)}{\sin(\phi)} \tag{43}$$

$$Z7=\frac{Fv*(L+a)*\tan(\phi)}{W+(L+a)*\tan(\phi)} \tag{44}$$

$$Z8=\frac{Fv*(W+c)*\tan(\phi)}{L+(W+c)*\tan(\phi)} \tag{45}$$

The volume of the soil pressure distribution in terms of Z5 is based on the sum of the volume of the six tetrahedrons listed here.

- 6, 4, 8, 5
- 3, 4, 5, 6
- 2, 3, 5, 6
- 2, 7, 5, 6
- 1, 3, 4, 6
- 1, 3, 2, 6

$$V=\frac{1}{6}*\left[\begin{bmatrix}0&0&Fv&1\\0&L&0&1\\0&L&Z8&1\\W&L&Z5&1\end{bmatrix}+\begin{bmatrix}W&L&0&1\\0&L&0&1\\W&L&Z5&1\\0&0&Fv&1\end{bmatrix}+\begin{bmatrix}W&0&0&1\\W&L&0&1\\W&L&Z5&1\\0&0&Fv&1\end{bmatrix}+\begin{bmatrix}W&0&0&1\\W&0&Z7&1\\W&L&Z5&1\\0&0&Fv&1\end{bmatrix}+\begin{bmatrix}0&0&0&1\\W&L&0&1\\0&L&0&1\\0&0&Fv&1\end{bmatrix}+\begin{bmatrix}0&0&0&1\\W&L&0&1\\W&0&0&1\\0&0&Fv&1\end{bmatrix}\right] \tag{46}$$

$$V=\frac{1}{6}*(W*L*Z8+W*L*Z5+W*L*Z5+W*L*Z7+W*L*Fv+W*L*Fv) \tag{47}$$

The center of gravity of the soil pressure volume shown here.

$$Xcg=\frac{1}{V*6}*\left(W*L*Z8*\frac{W}{4}+W*L*Z5*\frac{W}{4}+W*L*Z5*\frac{3*W}{4}+W*L*Z7*\frac{3*W}{4}+W*L*Fv*\frac{W}{4}+W*L*Fv*\frac{2*W}{4}\right) \tag{48}$$

$$Ycg=\frac{1}{V*6}*\left(W*L*Z8*\frac{3*L}{4}+W*L*Z5*\frac{3*L}{4}+W*L*Z5*\frac{2*L}{4}+W*L*Z7*\frac{L}{4}+W*L*Fv*\frac{2*L}{4}+W*L*Fv*\frac{L}{4}\right) \tag{49}$$

The footing eccentricity (e) is the perpendicular distance between the center of gravity of the soil pressure distribution and the center of the foundation. See equations 22 and 23.

4. Examples

4.1. Tower Footing

The input variables are as follows.

- V_f = 51000 lb   Footing vertical load
- W = 8 ft   Footing Width
- L = 6 ft   Footing Length
- MPL = 20000 ft-lb   Moment parallel
- MPR = 70000 ft-lb   Moment perpendicular

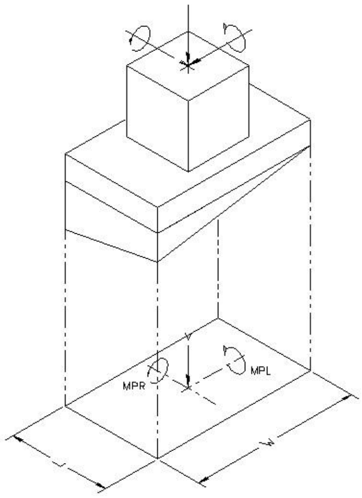

Figure 17. Example 4.1 tower footing

When maximum soil pressure is set to Fv = 2540 psf,

- From Equation 2, ϕ = 69 deg.
- From Equation 3, M = 72800 ft-lb
- From Equation 6, V1 = 5850 lb
- From Equation 10, V2 = 45100 lb
- From Equation 14, V3 = 609600 lb

Since the actual vertical load is 51000 then the characteristic shape is between the second and third transition volumes.

Iterate on Y3 between 0 and L until equation 35 equals input V_f. Y3 = 2.08 ft.

- From Equation 31, X4 = 6.51 ft-lb
- From Equation 32, Z7 = 230 psf
- From Equation 33, Z8 = 1880 psf
- From Equation 35, V = 51000 ft
- From Equation 36, Xcg = 2.58 ft
- From Equation 37, Ycg = 2.71 ft
- From Equation 22, e = 1.43 ft
- From Equation 23, M = 73000 ft-lb

If lower soil bearing pressure is assumed then there will not be sufficient eccentricity to balance the load. If higher soil bearing pressure is assumed then the eccentricity will be larger and there will be unused soil bearing and moment capacity.

Corner Coordinates

Point Number	X (ft)	Y (ft)	Z (lb/sq-ft)
1	0	0	0
2	8	0	0
3	8	2.08	0
4	6.50	6	0
5	0	6	0
6	0	0	2540
7	8	0	230
8	0	6	1880

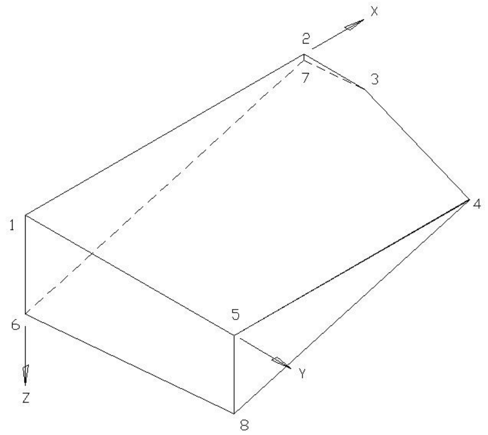

Figure 18. Example 4.1 soil pressure

4.2. Bridge Pier

This is a duplicate of an example from Young (1989) [5];

The input variables are as follows.

- V_f = 1,500 ton — Footing vertical load
- W = 20 ft — Footing Width
- L = 10 ft — Footing Length
- MPL = 4000 ft-ton — Moment parallel
- MPR = 4800 ft-ton — Moment perpendicular

When maximum soil pressure is set to Fv = 35.7 ton/sf,

- From Equation 2, ϕ = 31 deg.
- From Equation 3, M = 6248 ft-ton
- From Equation 6, V1 = 990 ton
- From Equation 10, V2 = 1420 ton
- From Equation 14, V3 = 3566 ton

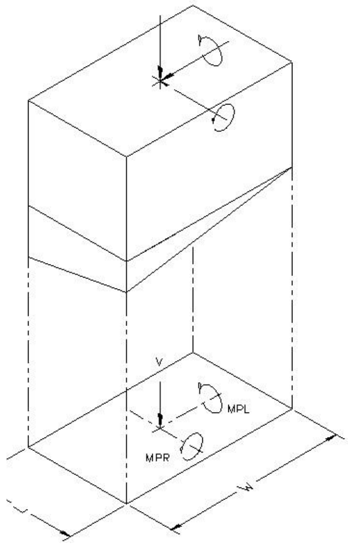

Figure 19. Example 4.2 bridge pier

With 1,500 ton vertical load the characteristic shape is between transition volume 2 and 3.

Iterate on Y3 between 0 and LW until equation 35 equals input V_f. Y3 = 0.35 ft.

- From Equation 31, X4 = 3.94 ft
- From Equation 32, Z7 = 1.0 ton/sf
- From Equation 33, Z8 = 6.84 ton
- From Equation 35, V = 1500 ton
- From Equation 36, Xcg = 5.18 ft
- From Equation 37, Ycg = 3.04 ft
- From Equation 22, e = 4.18 ft
- From Equation 23, M = 6270 ft-ton

If lower soil bearing pressure is assumed then there will not be sufficient eccentricity to balance the load. If higher soil bearing pressure is assumed then the eccentricity will be larger and there will be unused soil bearing and moment capacity.

Corner Coordinates			
Point Number	X (ft)	Y (ft)	Z (lb/sq-ft)
1	0	0	0
2	20	0	0
3	20	0.35	0
4	3.94	10	0
5	0	10	0
6	0	0	35.7
7	20	0	1.0
8	0	10	6.84

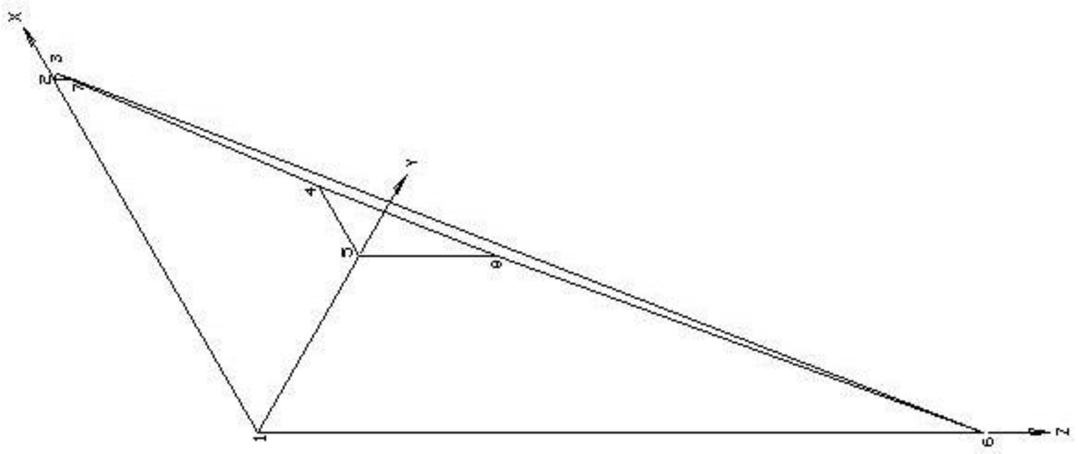

Figure 20. Example 4.2 Soil pressure

5. Conclusion

A straight forward and understandable extension of the uniaxial soil pressure calculation can be made for biaxial loading. Soil pressure distributions under footings subject to biaxial overturning can be calculated by modeling the soil pressure shape as an assembly of tetrahedrons and using matrix algebra to solve for the volume and center of gravity. These results are comparable with other solutions.

This method yields the corner coordinates of the pressure distribution and they can be input into solid modeling to confirm the volume's mass properties. The equations are presented in a form that can be automated to speed convergence on the solution.

6. Conflicts of Interest

The authors declare no conflict of interest.

7. References

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