



A Computational Approach to a Mathematical Model of Climate Change Using Heat Sources and Diffusion

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Abstract

The present work aims to extend the climate change energy balance models using a heat source. An ordinary differential equations (ODEs) model is extended to a partial differential equations (PDEs) model using the effects of diffusion over the spatial variable. In addition, numerical schemes are presented using the Taylor series expansions. For the climate change model in the form of ODEs, a comparison of the presented scheme is made with the existing Trapezoidal method. It is found that the presented scheme converges faster than the existing scheme. Also, the proposed scheme provides fewer errors than the existing scheme. The PDEs model is also solved with the presented scheme, and the results are displayed in the form of different graphs. The impact of the climate feedback parameter, the heat uptake parameter of the deep ocean, and the heat source parameter on global mean surface temperature and deep ocean temperature is also portrayed. In addition, these recently developed techniques exhibit a high level of predictability.

Keywords: Energy Balance Models; Heat Sources; Diffusion Effects; Numerical Scheme; Stability.

1. Introduction

In the 21st century, climate change is the most frequently faced global problem. The world's future is highly dependent upon the study of climatic changes and their consequences. For this purpose, global climate models are the best way to anticipate any change. Commonly, a climate model provides a mathematical representation of the atmosphere, oceans, land, and their physical, chemical, and biological principles. These principles provide the basis for deriving equations which are solved numerically over a grid using discrete steps in space and time. The time period could range from a few minutes to many years depending upon the requirement of the process under observation or capacity of computer programing and the choice of the numerical method.

Stability and scalability are the two main difficulties that climate researchers come across frequently. Stability refers to the stability of the solution with respect to initial conditions. However, scalability defines the increased resolution of current models. Nowadays, models with the highest resolution (mesoscale models) with too coarse a numerical grid exhibit difficulty in representing small-scale processes like turbulence in air and ocean boundary layers, interaction with small-scale topography features, thunderstorms, and cloud microphysics processes, etc. Fine-tuned approximations that were closely related to physical accuracy and computational viability were preferred by researchers.

Partial differential equations are the key equations used to build climate models. These are non-linear equations with the system, and their solutions are truly significant. One can gain the most general cases' existence, uniqueness, and

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stability by reducing difficulties through approximation and simplification. However, these equations may provide solutions to the problems like small variations in initial conditions give rise to divergent predictions.

Mathematics is a comprehensive branch of science that deals with problems and their prospective solutions. It resolves complex phenomena involved in engineering and technological sciences efficiently [1-4]; hence, one cannot deny its significance as a rationale for overcoming problems like the interaction between society and nature [5, 6]. Humans inhabiting a particular niche impart an adverse or beneficial impact on it and receive the aftermaths of floods, earthquakes, and pandemics due to disturbed ecological conditions. Nowadays, changing environmental conditions are human-induced making every species on earth more vulnerable [7, 8]. Global mean surface temperature (GMST) is the factor that is transformed by extreme climatic conditions. The constant surge in GMST led various experts to mull over the cost and effect of the consequences produced. A report [9] revealed that the average rise in GMST is $0.07^{\circ}\text{C}/\text{decade}$ since 1880. However, this value read to $0.17^{\circ}\text{C}/\text{decade}$ during the twenty-first century's first half, which is twice the average rate. The primary reason behind this surge is the excessive rate of carbon dioxide concentration in the atmosphere, which is produced from the combustion of fossil fuels. Excessive combustion increases the rate of carbon dioxide gas (also known as a greenhouse gas) in the atmosphere hence, acting as a major contributor to global warming.

Global warming altered climatic conditions substantially and differentially depending upon the locality [7, 8]. These changes affect the living community and its activities [10, 11]. Through specialized mathematical modeling, one can assess the climatic risk, which is an essential tool for evaluating the effects of environmental changes on humans [12]. These models contain input parameters such as temperature, humidity, and precipitation [13]. In contrast, climatic models have a variable degree of complexity for determining ecological changes on a regional and global level [14].

Recent estimates for regional and global climatic changes are characterized by uncertainties [15, 16]; however, Flato et al. (2014) [17], Grose et al. (2018) [18] and Colman & Soldatenko (2020) [19] provide a detailed comprehension of these causes. Therefore, it is vital to research the earth's climate system (ECS) under the effect of external heat sources and diffusion, either naturally or artificially, and to educate stakeholders about their surroundings. It is essential to prepare the communities for developing and implementing adaptation strategies and precautionary measurements necessary to dwindle the changing climatic effects. Adaptations are the measures taken by the population to adjust themselves to changing environments [7].

Approaches and mathematics procedures provide us with an opportunity to calculate and predict the effect of natural and artificial interference on the climatic system and its dynamics. Hence, one can say that this branch of science plays a significant role in climatic studies [20, 21]. Quantitative research is founded on mathematical models of observed systems and things. It is worth noting that in contrast to conventional problems of physics, reproducing and projecting climatic changes influenced by external forces possess individual properties which make them impossible to implement in full-scale direct physical experiments. Soldatenko (2017) [22] Revealed that the individuality of the environmental system made them difficult for laboratory experiments too. Moreover, the time series of climatic variables obtained from direct observations are insignificant for statistical calculations because these series are too short and have data for a few decades. Therefore, one can say that mathematical modeling is the only way to calculate the projection of the evolution of the climate system under natural and human interference.

2. Background

In late 1920, scientists began numerical studies on the climate, and the Norwegian Vilhelm Bjerknes developed a primitive equation around the turn of the century. These equations are the non-linear differential equations that reside on the principle of conservation of momentum and mass continuity duo with the thermal energy equation. The roots of the several atmospheric flow models were based on these equations. Bjerknes found the application of these models as weather anticipators in earlier 1900s. Bjerknes' interest was in fluid dynamics and electromagnetism. In his earlier years as an assistant in Bonn, he made significant contributions to Hertz's work based on electromagnetic resonance. Later in life, he joined the University of Bergen, a geophysical institution, where he published his book on the Dynamics of the Circular Vortex with Applications to the atmosphere in 1921. Carl-Gustaf Rossby, a pupil of Bjerknes in Bergen between 1919 and 1921, was a well-known Swedish meteorologist and the namesake of the Rossby Center.

The English scientist Lewis Fry Richardson took a great influence from Bjerknes' work and began to work numerically on meteorological equations. He substantially used the idea of discrete time-step in evaluating the equations. Later on, this idea was utilized by Courant, Friedrichs, and Lewy, a trio of mathematicians who developed a stability technique for Richardson-type numerical computations.

Fry Richardson's work produced unrealistic results of a single trial forecast allegedly due to errors in input data on the wind. His work did not provide a positive response from contemporary society but laid the ground for modern forecasting [23]. Several years after Fry Richardson's efforts, a plethora of developments took place in this field which gravitated more researchers. Advancements in numerical analysis allowed the application of stable algorithms possible. Additionally, meteorologists provide a clear understanding of atmospheric dynamics.

Moreover, the invention of radio sound and the improvement of the digital computer made climatic modeling a more diversified field in ecological studies. Therefore, world focus has shifted greatly towards algorithm and data science,

followed by climatic modeling. Furthermore, the need for meteorologists during World War II for weather anticipation raised, which ultimately surged their number twenty folds by the end of World War II and began the implementation of operational numerical weather prediction [24].

From 1946 to 1952, a renowned mathematician John von Neumann modified conventional computers to electronic construction and design at Princeton University. His work was considered revolutionary in the computer industry and consisted of four groups, of which one is devoted to meteorology. He recognized that a self-operated machine would be suited for weather computation after being influenced by the work of Courant, Friedrichs, and Lewy, as well as Fry Richardson's articles. John von Neumann straightened the path for exceptional research in meteorology in collaboration with the U.S navy department called Joint Numerical -Weather Prediction Unit (JN-WPU), chaired by meteorologist Jule Charney.

This group worked to draw the forecast of 24-hr by numerically solving a vortices equation and the total energy change [25]. The values were then compared and found closely related for the first 23 days. These simple calculations were considered a great achievement; however, they are insufficient as an accurate forecasting tool. This model demonstrated that skillful parameterization and sufficient processing power were required to develop a mathematical model of climate.

Energy balance models were studied in Lohmann (2020) [26] to provide admissible tools for understanding climate change. In the contribution [26], the global temperature was calculated. This temperature calculation is based on the earth's outgoing energy and incoming energy from the Sun. The zero-dimensional heat equation was considered in Lovejoy (2021) [27] to show that generally, conductive, radiative surface boundary condition leads to a half-order derivative relationship. The relationship is based on temperature and surface heat fluxes. Full inhomogeneous problem with thermal capacities, horizontal varying diffusivities, forcings and climate sensitivities. The 2D energy balance equations that can be used for climate projections, macro weather and studying the approach to new climate states were derived. For sea level rising risk management opportunities, a real options model has been developed [28]. A power penalty method was developed for the parabolic variational inequality reformulated from the problem. An energy balance model was studied in Vilar et al. (2021) [29] for an extensive green roof. The model represented the evolution of the temperature in two-layer. The layers were called vegetation and substrate layers. One characteristic of the model was it was heterogeneous.

Climate of today's world has been discorded due to anthropogenic activities which have produced serious drawbacks in the form of global warming and the greenhouse effect. Taking into account the hazardous effects of climatic change and uncertain events, one should be aware of the strategies and measures to keep themselves far from the evils of climatic harshness. For this, a person should have enough knowledge about climate and what measures must be adapted to mitigate its effects. Nowadays, mathematical modeling is considered the only tool to predict climatic evolution under natural and artificial involvement. During the past several decades, scientists used these tools and found them much more effective for the invention of ways to lessen the intensity of climate change.

The present work aims to construct numerical schemes for solving parabolic problems. The first stage of the scheme is explicit, while it is implicit in the second stage. The scheme provides fourth-order accuracy in time. For spatial discretization, any scheme can be considered. The scheme is constructed by employing Taylor series expansions; therefore, its accuracy can be proved by looking at the constructing procedure. The stability of the schemes depends on the parameter whose particular value gives a numerical scheme that can have a large stability region. Besides a scheme, an energy balance model is modified with source term(s). The model is solved using the proposed scheme.

One of the simple and effective climate system models is energy balance models (EBMs). These models were proposed about 50 years ago by Lohman (2020) [26] and Lovejoy (2021) [27]. These EBMs aimed to predict temperatures and how the climate system is sensitive to external forces. Since EBMs are linear, their exact solution can be found, so these models were constructed as a link between results gained by the complex models and theoretical concepts. In Chang et al. (2015) [28] work two boxes, EBM was considered under the effect of stochastic forcing parameterized by Gaussian white noise. The two-box EBM consists of global mean surface temperature T and deep ocean temperature T_D . These models can be considered to predict climate variation over several decades, which vary annually. The two-box EBM used in Chang et al. (2015) [28] is expressed as:

$$\frac{dT}{dt} = -\frac{\lambda}{C}T - \frac{\gamma}{C}(T - T_D) + F, \quad (1)$$

$$\frac{dT_D}{dt} = \frac{\gamma}{C_D}(T - T_D), \quad (2)$$

where C denotes the effective heat capacity for the upper box, λ denotes the climate feedback parameter, γ represents the heat uptake parameter of the deep ocean, C_D represents the effective heat capacity of the bottom box or deep ocean, and F represents the stochastic forcing. Here instead of two ordinary differential equations with one independent variable time, a climate model of partial differential equations is considered with the effect of diffusion and a non-linear

source term that increases the mean surface temperature and deep-ocean temperature. If the coefficient of diffusion and non-linear force term approaches zero, the model becomes a simple EBM with zero force. The partial differential equations (PDEs) model also describes the variation of climate over spatial variables. The PDEs model for climate system can be expressed as:

$$\frac{\partial T}{\partial t} = d_1 \frac{\partial^2 T}{\partial x^2} - \frac{\lambda}{c} T - \frac{\gamma}{c} (T - T_D) + \frac{QT^2}{c} \tag{3}$$

$$\frac{\partial T_D}{\partial t} = d_2 \frac{\partial^2 T_D}{\partial x^2} + \frac{\gamma}{c_D} (T - T_D) + \frac{Q_1}{c_D} T \tag{4}$$

Before starting the construction procedure of the numerical scheme, a stability analysis of the model with $d_1 = d_2 = 0$ will be presented. For doing so, the equilibrium points for the model have an only ordinary differential equation with (with $d_1 = d_2 = 0$ in Equations 3 and 4) will be determined. So consider the following equations:

$$0 = -\frac{\lambda}{c} T - \frac{\gamma}{c} (T - T_D) + \frac{QT^2}{c} \tag{5}$$

$$0 = \frac{\gamma}{c_D} (T - T_D) + \frac{Q_1}{c_D} T \tag{6}$$

The equilibrium points are expressed as:

$$B_1(0,0), B_2\left(\frac{\lambda-Q_1}{Q}, \frac{(\lambda-Q_1)(\gamma+Q_1)}{\gamma Q}\right) \tag{7}$$

To find the system's stability, the set of ordinary differential equations from Equations 3 and 4 will be linearized.

At the non-zero equilibrium point, the Jacobian can be expressed as:

$$J|_{B_2} = \begin{bmatrix} -\gamma - \lambda + 2(\lambda - Q_1) & \gamma \\ \gamma + Q_1 & -\gamma \end{bmatrix} \tag{8}$$

The characteristic polynomial for $J|_{B_2}$ is expressed as:

$$P(x) = -\gamma\lambda + \gamma Q_1 + 2\gamma x - \lambda x + 2Q_1 x + x^2 \tag{9}$$

Using the Routh-Hurwitz criterion for second-degree polynomial, the eigenvalues will be negative if $a_0, a_1 > 0$ for the characteristic polynomial $a_0 + a_1 x + x^2 = 0$. For this case, the eigenvalues of $J|_{B_2}$ will be negative if and only if:

$$-\gamma\lambda + \gamma Q_1 > 0 \text{ and } 2\gamma - \lambda + 2Q > 0 \tag{10}$$

3. Proposed Numerical Scheme

This study also constructs a numerical scheme for solving time-dependent partial differential equations. For doing this, a difference equation will be given for finding a solution at some time level [30-32]. This stage will be called a first stage of the presented scheme. The first stage of the scheme is expressed as:

$$\bar{T}_i^{n+1} = T_i^n + \Delta t \left(\frac{\partial T}{\partial t}\right)_i^n \tag{11}$$

where \bar{T}_i^{n+1} is unknown at some time level. For the second stage, a difference equation with some unknown parameters is expressed as:

$$T_i^{n+1} = aT_i^n + b\bar{T}_i^{n+1} + \Delta t \left\{ c \left(\frac{\partial T}{\partial t}\right)_i^{n+1} + e \left(\frac{\partial \bar{T}}{\partial t}\right)_i^{n+1} + f \left(\frac{\partial T}{\partial t}\right)_i^n \right\} \tag{12}$$

The Taylor series expansions for T_i^{n+1} and $\left(\frac{\partial T}{\partial t}\right)_i^n$ are given as:

$$T_i^{n+1} = T_i^n + \Delta t \left(\frac{\partial T}{\partial t}\right)_i^n + \frac{(\Delta t)^2}{2} \left(\frac{\partial^2 T}{\partial t^2}\right)_i^n + \frac{(\Delta t)^3}{6} \left(\frac{\partial^3 T}{\partial t^3}\right)_i^n + O((\Delta t)^4) \tag{13}$$

$$\left(\frac{\partial T}{\partial t}\right)_i^{n+1} = \left(\frac{\partial T}{\partial t}\right)_i^n + \Delta t \left(\frac{\partial^2 T}{\partial t^2}\right)_i^n + \frac{(\Delta t)^2}{2} \left(\frac{\partial^3 T}{\partial t^3}\right)_i^n + O((\Delta t)^3) \tag{14}$$

$$\left(\frac{\partial \bar{T}}{\partial t}\right)_i^{n+1} = \left(\frac{\partial T}{\partial t}\right)_i^n + \Delta t \left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \tag{15}$$

Upon substituting Equations 11 and 13 to 15 into Equation 12 and comparing coefficients of $T_i^n, \left(\frac{\partial T}{\partial t}\right)_i^n, \left(\frac{\partial^2 T}{\partial t^2}\right)_i^n$ and $\left(\frac{\partial^3 T}{\partial t^3}\right)_i^n$ on both sides of the resulting equation, gives the following numerical values and expressions:

$$\left. \begin{aligned} b &= 1 - a \\ c &= \frac{1}{3} \\ e &= \frac{1}{6} \\ f &= \frac{-1+2a}{2} \end{aligned} \right\} \tag{16}$$

Substituting Equation 1 into Equation 12, it is obtained:

$$T_i^{n+1} = aT_i^n + (1 - a)\bar{T}_i^{n+1} + \Delta t \left\{ \frac{1}{3} \left(\frac{\partial T}{\partial t} \right)_i^{n+1} + \frac{1}{6} \left(\frac{\partial \bar{T}}{\partial t} \right)_i^{n+1} + \left(\frac{-1+2a}{2} \right) \left(\frac{\partial T}{\partial t} \right)_i^n \right\} \tag{17}$$

An iterative scheme will be employed for solving Equations 17. So, in this manner solution of Equations 3 and 4 with some boundary and initial conditions can be found.

4. Stability Analysis

It is well known from the literature that the stability of finite difference schemes can be checked by employing the von Neumann stability criterion. For the present case, the system of partial differential equations is considered, so for the stability of this system. A matrix-vector equation will be constructed as

$$\frac{\partial \phi}{\partial t} = A \frac{\partial^2 \phi}{\partial x^2} \tag{18}$$

where $A = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$, $\phi = \begin{bmatrix} T \\ T_D \end{bmatrix} - B_2$.

Before beginning the procedure of finding stability condition, Equation 18 is discretized by the presented scheme. So, the first stage of the scheme for discretization of Equation 18 is expressed as:

$$\bar{\phi}_i^{n+1} = \phi_i^n + \Delta t A \delta_x^2 \phi_i^n \tag{19}$$

and the second stage of the scheme for discretization of Equation 18 can be expressed as:

$$\phi_i^{n+1} = a\phi_i^n + (1 - a)\bar{\phi}_i^{n+1} + \Delta t \left\{ \frac{1}{3} (A \delta_x^2 \phi_i^{n+1}) + \frac{1}{6} (A \delta_x^2 \bar{\phi}_i^{n+1}) + \left(\frac{-1+2a}{2} \right) (A \delta_x^2 \phi_i^n) \right\} \tag{20}$$

For employing the von Neumann stability criterion, components of the Fourier series are given as:

$$\left. \begin{aligned} \bar{\phi}_i^{n+1} &= \bar{E}^{n+1} e^{iI\theta}, \phi_i^{n+1} = E^{n+1} e^{iI\theta} \\ \phi_{i\pm 1}^n &= E^n e^{(i\pm 1)I\theta}, \phi_{i\pm 1}^{n+1} = E^{n+1} e^{(i\pm 1)I\theta} \end{aligned} \right\} \tag{21}$$

where E^{n+1} is an amplitude of the wave at $(n + 1)th$ time level, θ is a wave number and $I = \sqrt{-1}$.

Substituting transformations from Equation 21 into Equation 19 and dividing by $e^{iI\theta}$ on both sides of the resulting equation, it gives:

$$\bar{E}^{n+1} = E^n + \Delta t \left\{ A \left(\frac{2\cos\theta - 2}{(\Delta x)^2} \right) E^n \right\} = (1 + 2d(\cos\theta - 1))E^n \tag{22}$$

where $d = \frac{\Delta t}{(\Delta x)^2}$.

Similarly, for the second stage, substituting transformations from Equations (21) into Equation 20, it yields:

$$E^{n+1} = aE^n + (1 - a)\bar{E}^{n+1} + \Delta t \left\{ \frac{1}{3} \left(\frac{2A(\cos\theta - 1)}{(\Delta x)^2} \right) E^{n+1} + \frac{1}{6} \left(\frac{2A(\cos\theta - 1)}{(\Delta x)^2} \right) (1 + 2dA(\cos\theta - 1))E^n + \left(\frac{-1+2a}{2} \right) \left(\frac{2A(\cos\theta - 1)}{(\Delta x)^2} \right) E^n \right\} \tag{23}$$

The amplification factor can be expressed as:

$$\frac{E^{n+1}}{E^n} = \left\{ a + (1 - a)(1 + 2dA(\cos\theta - 1)) + \frac{d}{3} A(\cos\theta - 1)(1 + 2dA(\cos\theta - 1)) + (-1 + 2a)dA(\cos\theta - 1) \right\} / \left\{ 1 - \frac{2}{3} dA(\cos\theta) \right\} \tag{24}$$

where $d = \frac{\Delta t}{(\Delta x)^2}$.

The stability condition for the proposed scheme is given as:

$$\left| \frac{E^{n+1}}{E^n} \right| \leq 1 \tag{25}$$

5. Results and Discussions

The presented numerical scheme is fourth-order accurate, and its accuracy can be proved by applying the Taylor series expansions. Since the scheme has been developed using the Taylor series expansion, it is fourth-order in time from its developing procedure. Also, the proposed scheme is consistent and can be verified using Taylor series expansions. Since every scheme is at least first-order accurate is consistent, the proposed scheme is also consistent.

The flow chart of the presented research is given in Figure 1. Figure 2 shows the comparison of two schemes over the variable t . From this Figure 2, it is clear that the proposed scheme performs better than the existing trapezoidal method for chosen values of parameters. Since Figure 2 is drawn for just the ODEs model, Matlab solver ode45 can be employed to find the considered model's solution without diffusion effects. Figure 2 is drawn using norms of difference of proposed/trapezoidal schemes and Matlab solver ode45. Also, since the proposed and existing schemes are implicit, iterative methods are employed to solve the difference equations obtained by both schemes. Once an iterative procedure is started, the solution will be found on each iteration, and this iterative procedure will be stopped if the given criteria are met. So, in this manner, the scheme will converge fast if it consumes lesser iterations than those consumed by another scheme. Figure 3 shows the behavior of global mean surface temperature T and deep ocean temperature T_D by varying climate feedback parameter λ . Global mean surface temperature and deep ocean temperature decrease by growing climate feedback parameter over t , but the opposite trend can be seen over the spatial variable x . Figure 4 shows the impact of the heat uptake parameter of the deep ocean on global mean surface temperature and deep ocean temperature. Both types of temperatures de-escalate over t by growing values of heat uptake parameter. The variation of heat uptake parameter on global mean surface temperature is very small, and both increasing and decreasing behaviors of deep ocean temperature over spatial variable x can be seen in this Figure 4. Figure 5 shows the impact of heat source parameter Q on the global mean surface and deep ocean temperature. Both types of temperatures escalate by growing heat source parameter Q over the variable t . Similar temperature behavior over spatial variable x can be seen in Figure 5 when the heat source parameter is enhanced. Figures 6 to 9 show the contour plots of global mean surface and deep ocean temperature with varying values of the second heat source parameter.

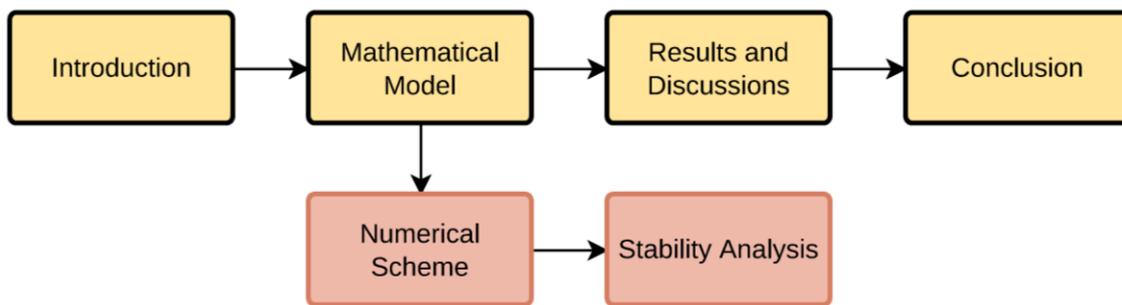


Figure 1. Flow chart of the presented research

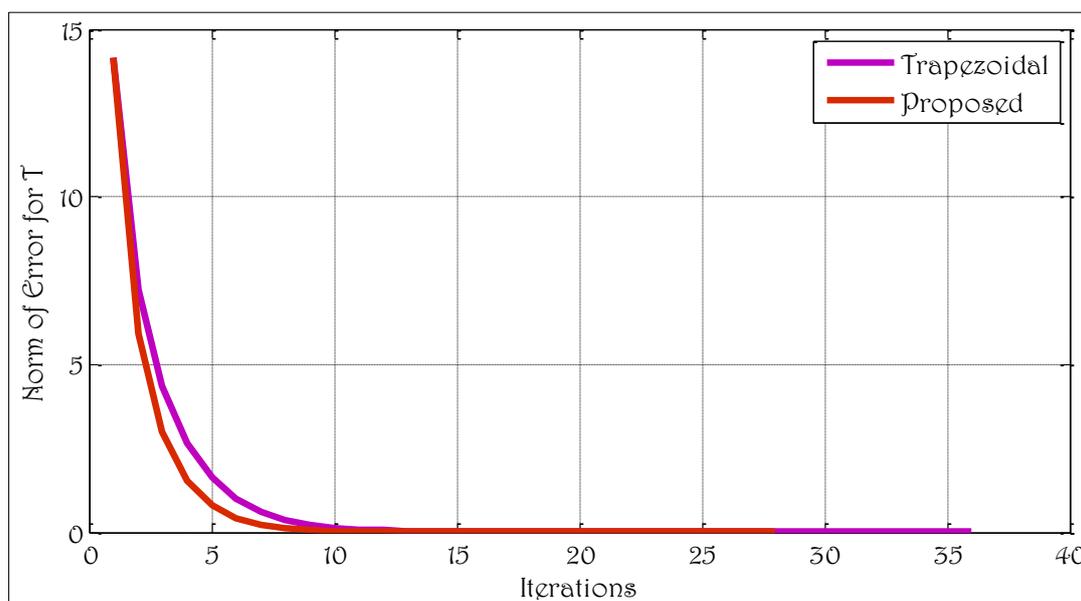


Figure 2. Convergence of two schemes using $\lambda = 0.5, \gamma = 0.7, Q = 0.017, Q_1 = 0.07, d_1 = 0, d_2 = 0, N_t = 900, u_0 = 4, v_0 = 0$

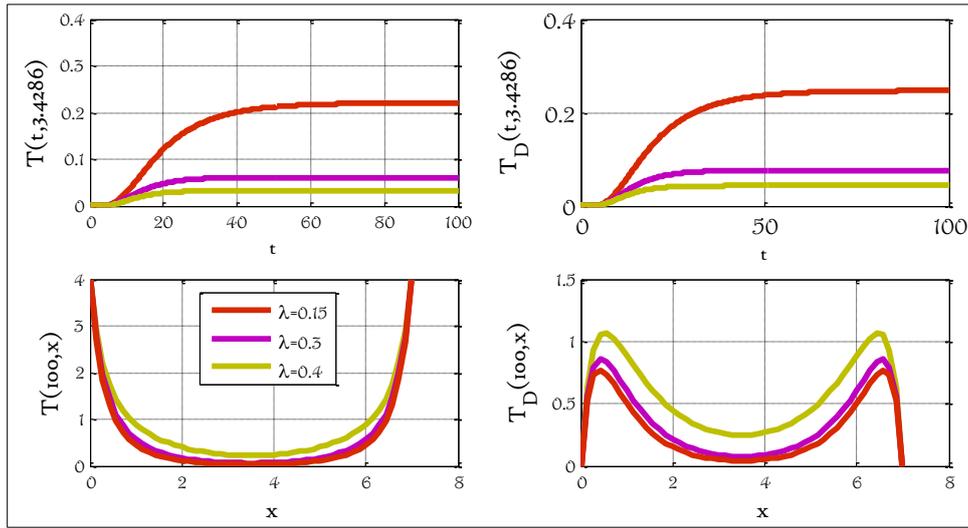


Figure 3. Impact of climate feedback parameter on global mean surface temperature and deep ocean temperature using $\gamma = 0.7, Q = 0.01, Q_1 = 0.01, d_1 = 0.1, d_2 = 0.1, N_x = 50, N_t = 500$

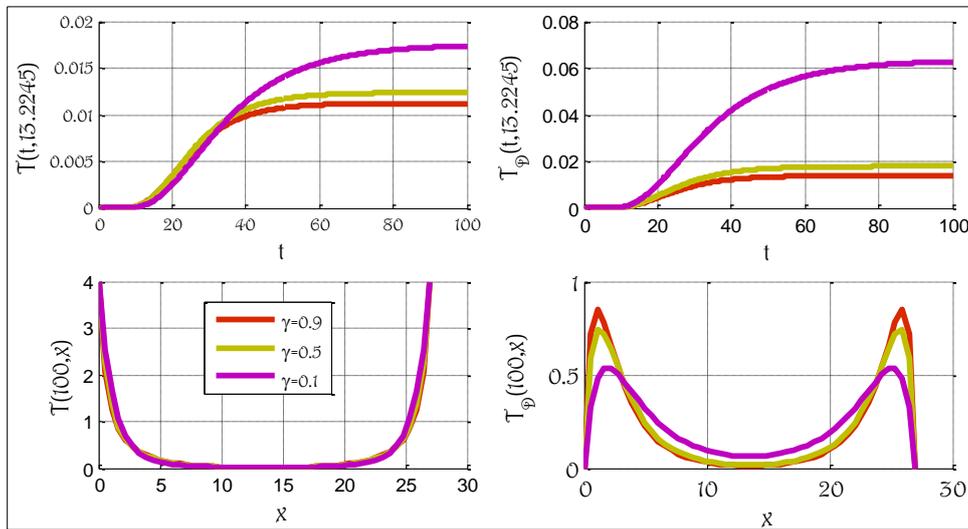


Figure 4. Impact of heat uptake parameter of the deep ocean on global mean surface temperature and deep ocean temperature using $\lambda = 0.3, Q = 0.017, Q_1 = 0.07, d_1 = 0.5, d_2 = 0.7, N_x = 50, N_t = 900$

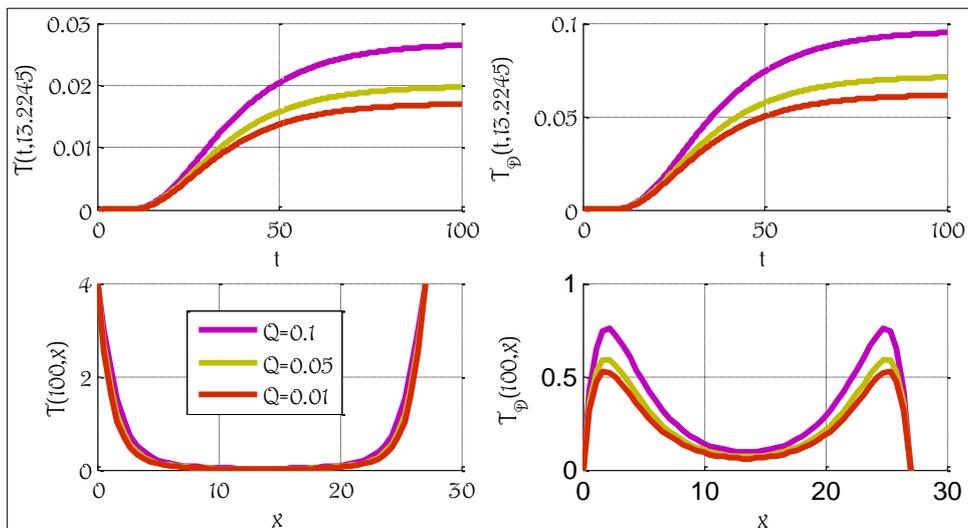


Figure 5. Impact of heat source parameter on global mean surface temperature and deep ocean temperature using $\lambda = 0.3, \gamma = 0.1, Q_1 = 0.07, d_1 = 0.5, d_2 = 0.7, N_x = 50, N_t = 900$

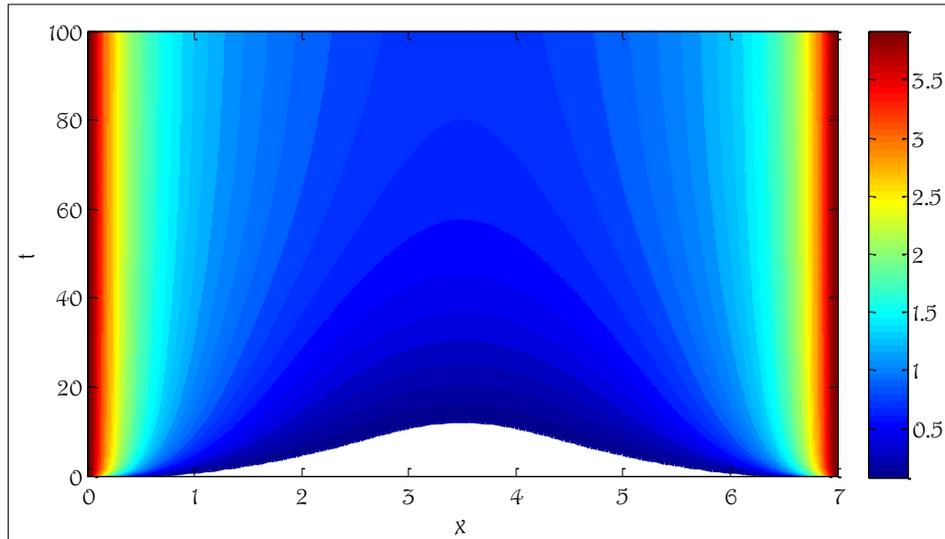


Figure 6. Contour plot of global mean surface temperature using $\lambda = 0.15, \gamma = 0.7, Q = 0.01, Q_1 = 0.1, d_1 = 0.1, d_2 = 0.1, N_x = 50, N_t = 500$

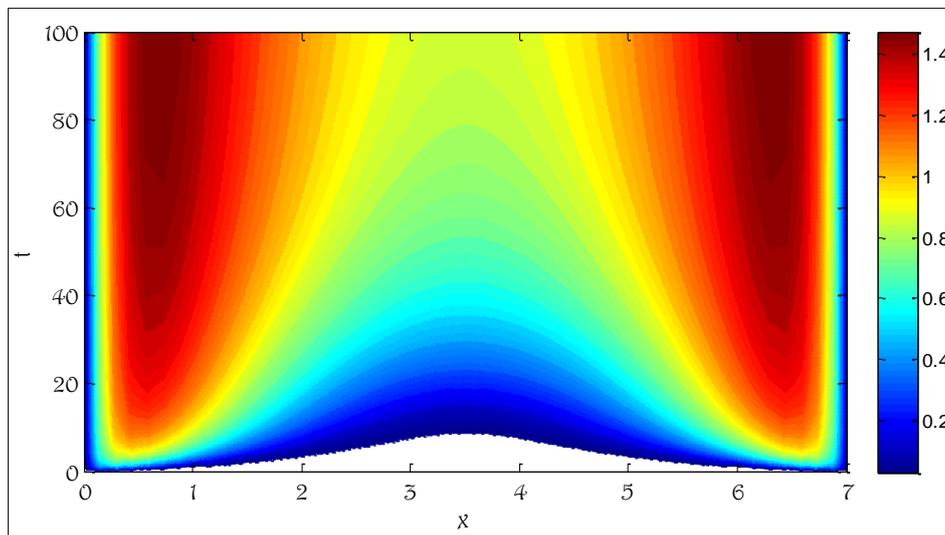


Figure 7. Contour plot of deep ocean temperature using $\lambda = 0.15, \gamma = 0.7, Q = 0.01, Q_1 = 0.1, d_1 = 0.1, d_2 = 0.1, N_x = 50, N_t = 500$

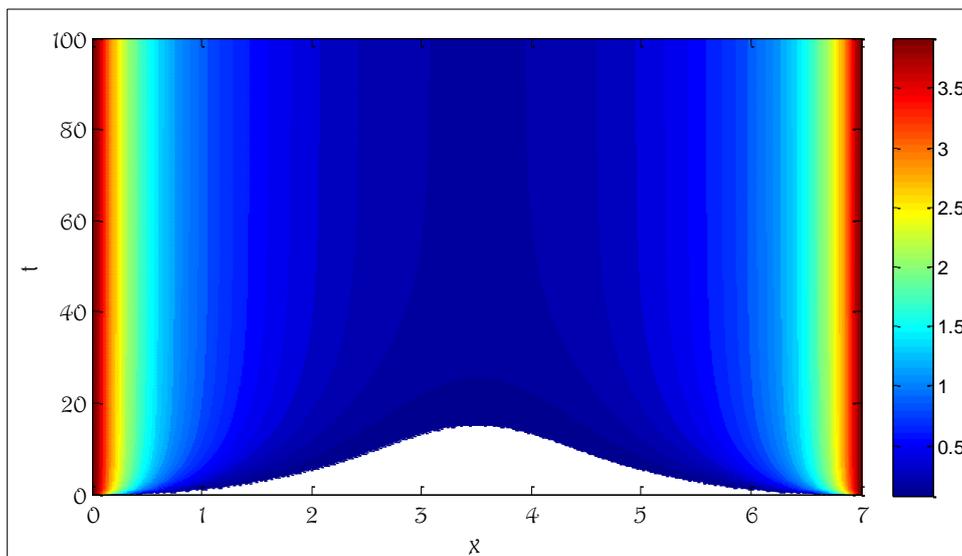


Figure 8. Contour plot of the global mean surface temperature using $\lambda = 0.15, \gamma = 0.7, Q = 0.01, Q_1 = 0.01, d_1 = 0.1, d_2 = 0.1, N_x = 50, N_t = 500$

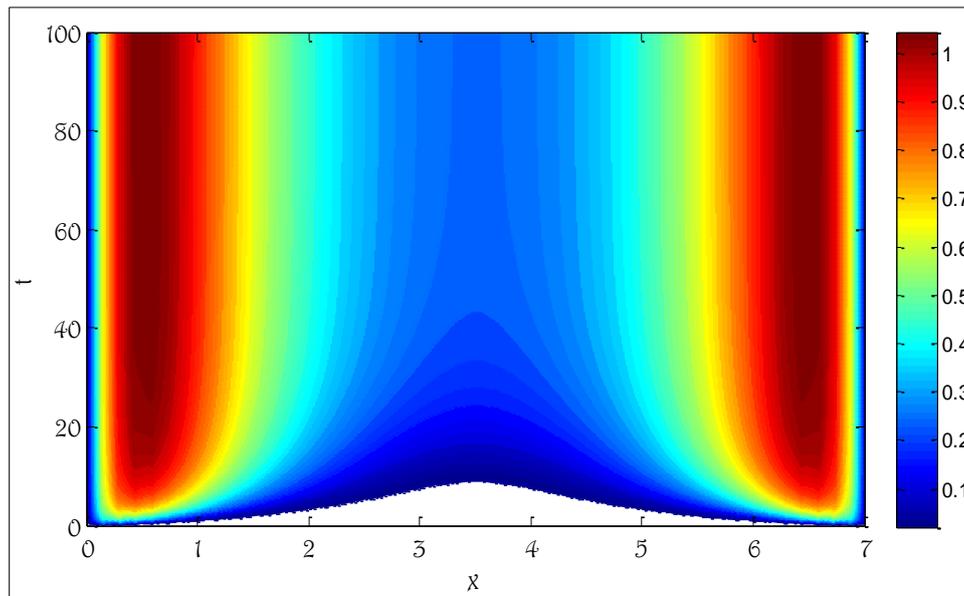


Figure 9. Contour plot of deep ocean temperature using $\lambda = 0.15, \gamma = 0.7, Q = 0.01, Q_1 = 0.01, d_1 = 0.1, d_2 = 0.1, N_x = 50, N_t = 500$

Table 1 shows the comparison of the proposed scheme with the trapezoidal method. The trapezoidal method gives second-order accuracy, while the proposed scheme provides third-order accuracy. The difference between the two numerical solutions is shown in the table. Since the model's exact solution is unknown, the numerical solution obtained by Matlab solver ODE45 is considered in place of the exact solution. The norm of error produced by the proposed scheme is less than that obtained by the existing trapezoidal method. But, the norm of error for the Trapezoidal method decreases by increasing the number of nodes, but for the proposed scheme, the norm is reduced and then enhanced by increasing the number of nodes.

Table 1. Comparison of the proposed scheme with the Trapezoidal method

No. of Nodes (N)	$\ bvp4c - Trapezoidal\ _2$	$\ bvp4c - Proposed\ _2$
500	0.0161	0.0024
700	0.0097	0.0019
900	0.0067	0.0020

The limitations of the proposed schemes are their conditional stability and applicability to every type of partial differential equations (PDEs). Most of the schemes are conditionally stable, so the step size in space and time should be chosen appropriately to obtain a convergent solution.

6. Conclusion

Two kinds of extensions have been provided for energy balance models in climate change. The use of heat sources and spatial diffusion effects have also been incorporated. A numerical scheme has been presented for solving both ODEs and PDEs models. A comparison has also been made with the existing scheme. For comparison, a Matlab solver ode45 has been utilized. This Matlab solver was employed to solve the ODEs model with chosen values of parameters. From the displayed results, it has been found that global mean surface temperature and deep ocean temperature decreased by an increment in the deep ocean's climate feedback parameter and heat uptake parameter. But the opposite behavior has been observed with the growing values of heat source parameters over time. This study will provide a detailed elaboration of a climatic model with respect to partial differential equations and a brief introduction containing the historical background of the environmental study, followed by the description of the Navier-Stokes equation. This equation has laid the foundation of environmental studies, hence, it is considered substantial. The current study is cemented by the fact-based popular approximations of the surveys and modeling techniques, which were the focus of several researchers for thousands of decades. The numerical approaches that have been provided are versatile enough to be applied to a variety of time-dependent problems whose spatial derivative terms can be of either integer or q th order. In addition, the proposed numerical approach has the potential to be utilized in epidemiological illness models as well as other types of issues in fractional calculus [33, 34]. The limitation of most of the proposed schemes is their conditional stability. In the future, numerical schemes can be constructed with larger stability regions.

7. Declarations

7.1. Author Contributions

Conceptualization, M.S.A. and Y.N.; methodology, M.S.A.; software, Y.N.; validation, Y.N., M.S.A. and K.A.; formal analysis, Y.N.; investigation, M.S.A.; resources, K.A.; data curation, K.A.; writing—original draft preparation, M.S.A.; writing—review and editing, Y.N.; visualization, K.A.; supervision, M.S.A.; project administration, K.A.; funding acquisition, K.A. All authors have read and agreed to the published version of the manuscript.

7.2. Data Availability Statement

The data presented in this study are available on request from the corresponding author.

7.3. Funding

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7.5. Conflicts of Interest

The authors declare no conflict of interest.

8. References

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