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Analysis of Combined Vertical and Radial Consolidation of Soil under Time-Dependent Loading

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Abstract

Fine-grained compressible soils are usually associated with strength challenges in the course of carrying structures, roadways, embankments, and other civil engineering facilities. In addition, due to their low permeabilities, compressible soils take an awfully long duration to achieve optimal consolidation, with its attendant negative effects on the facilities supported by the soils. Engineering practitioners and researchers have established the efficacy of vertical drains for accelerating the consolidation process of such soils through the shortening of horizontal flow distances, thereby stabilizing them and improving their load-bearing capacities. Application of the pre-loading surcharge provides additional drive for rapid consolidation. The case of soils carrying time-dependent loading is quite topical because it reflects reality most appropriately. However, a rigorous analysis of soils undergoing vertical and radial consolidation with a constant or time-varying surcharge is conspicuously lacking in the literature because most authors of publications in this subject area have largely based their solution procedure on the assumed decoupling of the vertical and radial flows by treating their associated pore pressures as separate. This assumption, notwithstanding the simplification it introduces into the mathematics of the problem, is not supported by physics. Therefore, the theory presented herein aims at addressing that gap in the literature. Throughout this analysis, the coupled (vertical and radial) flow, driven by a common pore water pressure, is handled as a single process. Successive applications of the integral transformations of Laplace and finite Hankel have been used to obtain an explicit expression for the image of the pore water pressure as a function of the transformation parameters. This is followed by successive inversions of the integral transforms, leading to a closed-form solution in the sense of a generalized Fourier series. The classical definition of the average degree of consolidation is easily applied in this case, unlike other methods in the literature that rely on the principle of superposition, whose applicability in this circumstance remains questionable. The validity of the present analysis has been established through logical checks and comparison with previous results in the literature. This theory has been proven to be applicable to cases of constant as well as time-varying surcharges.

Keywords: Consolidation; Integral Transform; Laplace; Finite Hankel; Time-Dependent Loading; Vertical and Radial Drains.

1. Introduction

Fine-grained compressible soils are usually associated with strength challenges, in the course of supporting structures, roadways, embankments and other civil engineering facilities. In addition, such soils, when loaded in their natural states would normally undergo very slow consolidation process (sometimes running up to twenty years) due to low permeabilities of clays [1]. After such a long period, problems associated with consolidation including settlement cracks in engineering structures, and failures in roads and airfields become noticeable and sometimes difficult to correct. For over decades, researchers and practitioners in the field of geotechnical engineering have invested intellectual

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resources in addressing these challenges, especially in devising practical ways of stabilizing such soils in order to improve their load carrying characteristics and other engineering properties The use of vertical drains to accelerate soil consolidation process has proved to be a most outstanding solution as regards improvement of the engineering properties of compressive soils [2, 3].

Further improvement on this approach is the introduction of a pre-loading surcharge, that is, the provision of external loading on the soil to act as a facilitator for the consolidation process. A variant of this model is to consider the surcharge loading as time-dependent [4]. This time-dependent surcharge model is indeed closer to reality than the so-called instantaneous loading in the sense that it captures the load build-up phase due to construction culminating in the ultimate load level in cases where the actual structure serves as the consolidating surcharge. Consistent with the practice of engineering, analytical devices became necessary to be developed for accurate evaluation of the degree of consolidation and other relevant parameters for soils undergoing both vertical and radial consolidation under time-dependent surcharge.

Perhaps, Barron [5] is the most prominent among the pioneer works on the analysis of consolidation of soils with the use of vertical sand drains. Barron's analytical model rests on the hypothesis that the associated vertical strain consists of the "free vertical strain" and the "equal vertical strain". The former assumes that the surface load is uniformly distributed and that the differential settlements over the domain of influence of each well, as consolidation progresses, have no effect on the distribution of stresses due to the arching of the fill. The latter assumes that horizontal sections remain horizontal throughout the process of consolidation [2, 6]. Arguably, the equal vertical strain hypothesis may have some accuracy advantage over the free vertical strain. However, the complications encountered in the application of the equal vertical strain approach may have given rise to the preference of choice in favour of the free vertical strain hypothesis [3]. The present consideration shall therefore be based on the free vertical strain hypothesis.

Presently, the literature on the subject of vertical and radial soil consolidation with constant or time-varying surcharge relies on the concept that the pore water pressure that drives the consolidation process has two components, namely: (a) the pore water pressure responsible for the radial flow and (b) the pore pressure for the vertical flow. On the strength of this separation, Carrillo [7] proposed that the average degree of consolidation for the soil should be obtained through the superposition of the separate degrees of consolidation associated with the radial and vertical flows. Olson [8] subsequently applied the Carrillo [7] approach in solving the consolidation problem of a soil with vertical drain under a ramp surcharge on the basis of equal strain assumption. A number of researchers have made presentations on vertical and radial consolidation with average degrees of consolidation computed on the Carrillo model [1, 9-11]. However, other researchers have advanced reasons for considering Carrillo's concept as inaccurate, with respect to the evaluation of the degree of consolidation as regards time-dependent surcharge on the soil [12-15]. Notwithstanding, the literature still lacks proven analytical tools for obtaining better results than the Carrillo approach [7].

It is the considered opinion of the author of this presentation that separating the radial flow pore pressure from that of the vertical flow, akin to decoupling of the flows, as proposed by Carrillo [7] lacks justification in physics. Therefore, development of a rigorous theory, capable of determining the single pore water pressure responsible for both flows, in their coupled configuration, is still a compelling necessity. Consequently, this presentation aims at solving the governing differential equation for the coupled radial and vertical flows as a single process, for the purpose of addressing the identified gap in the literature. Achievement of this aim rests upon successive applications of the Laplace, and finite Hankel transformations yielding the image of the dependent variable (the single pore water pressure for both flows) as a function of the integral transform parameters. Subsequent inversion of the transformations leads to a closed-form solution for the pore water pressure as a function of space and time, in the form of generalised Fourier series. The average degree of consolidation of the soil is obtained directly, without any form of superposition. The method advanced in this presentation has been subjected to certain validation checks, including recovery of the Terzaghi one-dimensional consolidation solution. A numerical illustration of the method in this presentation yielded results that compared very well with results from previous works reported by other researchers [16].

Indeed, the presentation made herein can easily be simplified to handle one-dimensional soil consolidation under time-dependent surcharge. It is pertinent to note that most of the earlier works in the research area of consolidation assumed instantaneous application of the loading on the soil undergoing consolidation [17]. While this assumption largely simplifies the problem of analysing the consolidation process, it fails to give a reliable representation of the practical field situation. To address this observation, Terzaghi [6] put forward an argument that it was necessary to account for the time-dependence in the build-up of the loading during the construction period, a situation that necessitated Terzaghi's introduction of some correction in the time-settlement curve under instantaneous loading for his one-dimensional consolidation theory [7]. However, Terzaghi's correction is based on empirical rather than analytical foundation. A number of researchers have attempted to incorporate the time-dependence of the loading process in solving the consolidation problem. As the awareness that consolidation process under instantaneous loading has inherent inaccuracies grows, researchers have started paying greater attention to time-variation in the loading of soils for both one-dimensional and three-dimensional consolidation processes [6, 8, 12-14, 18, 19].

It is worthy of note that results of various numerical and experimental methods on the characterisation of soils undergoing vertical and radial consolidation, with constant or time-varying surcharge, have been reported in the literature [20-24]. However, a rigorous theory to benchmark the accuracy of those results is so far lacking. The presentation by this author will serve to define the accuracy levels of such publications and future ones.

Against the backdrop of the foregoing facts, it is easily seen that the strength of this presentation lies in the fact that it has chosen a completely different route in solving the partial differential equation for the vertical and radial consolidation process. The selected route, as captured pictorially in Figure 1, has the following as its major features.

- Presentation of a rigorous analytical methodology for solving the problem of combined vertical and radial consolidation of soils under time-dependent loading as a single process without decoupling the vertical and radial flows;
- The development of a solution procedure that yields analytical (closed-form) expression for the excess pore pressure, in the sense of generalized double Fourier series; and
- Notwithstanding the single ramp-loading surcharge used in this presentation, application of successive integral transforms (Laplace and Finite Hankel) is sufficiently versatile to handle other time-dependent load models.

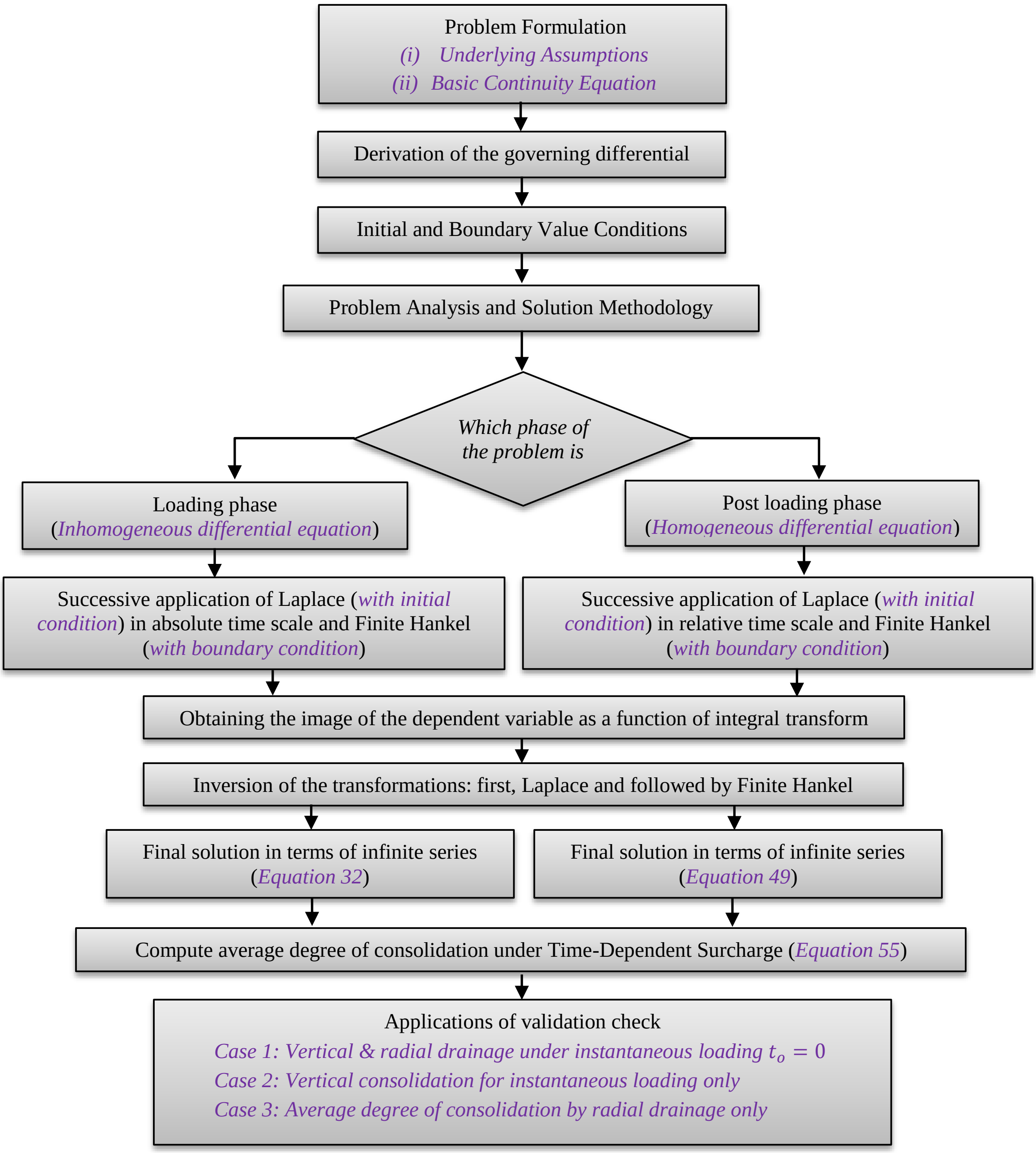

Figure 1. Methodology flowchart

2. Problem Formulation

This presentation is based on the following assumptions, which are consistent with Barron's and indeed Terzaghi's principles in founding the consolidation theory.

- The soil under investigation is fully saturated.
- Both the water and the soil particles have negligible compressibility.
- Darcy's law holds.
- All compressive strains within the soil mass are small and they occur in the vertical direction only, implying that compression is confined and shear strains are ignored.
- The field of influence of each well is a circular cylinder, see Figure 2 below.
- The loading on the surface is uniformly distributed.
- The coefficient of compressibility is constant throughout the consolidation process.
- Barron's free vertical strain hypothesis is valid.
- The consolidating soil has pervious top and impervious bottom.
- Smear effects and other resistances on the consolidation process are ignored

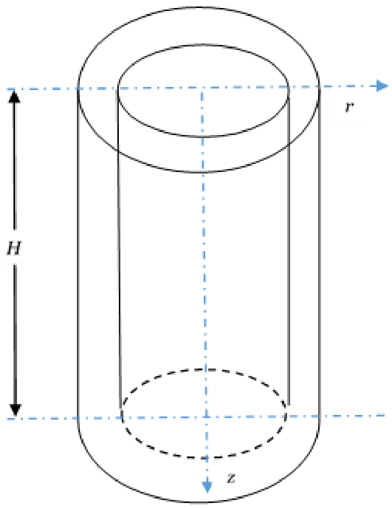

Figure 2. A typical drainage well

Now, in line with the derivations made by Leo [16], the consolidation process takes off from the following equation of continuity:

$$\frac{k_r}{\gamma_w}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+\frac{k_v}{\gamma_w}\frac{\partial^2 u}{\partial z^2}=-\frac{\partial \mathcal{E}_z}{\partial t} \quad (1)$$

where k_r and k_v are the radial and vertical permeabilities respectively, γ_w is the unit weight of water, u is the excess pore water pressure, r and z are the spatial coordinates, and t is the temporal coordinate, , $\mathcal{E}_z$ is the compressive strain (which is in the vertical direction) and is given by:

$$\mathcal{E}_z = m_V \sigma_z' \quad (2)$$

where m_V is the coefficient of volume compressibility and σ_z' is the effective stress in the z-direction. Furthermore,

$$\sigma_z' = q(t) - u \quad (3)$$

$q(t)$ being the total pressure, which in this case is the surcharge. Substituting from Equation 3 into Equation 2, we easily obtain

$$\mathcal{E}_z = m_V\{q(t) - u\} \quad (4)$$

It follows from Equations 1 and 4 therefore that:

$$\frac{k_r}{m_V\gamma_w}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+\frac{k_v}{m_V\gamma_w}\frac{\partial^2 u}{\partial z^2}=-\frac{\partial q}{\partial t}+\frac{\partial u}{\partial t} \tag{5}$$

Equation 5 can be re-arranged to read as follows:

$$C_r\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+C_v\frac{\partial^2 u}{\partial z^2}-\frac{\partial u}{\partial t}=-\frac{\partial q}{\partial t} \tag{6}$$

where $C_r=\left(\frac{k_r}{m_V\gamma_w}\right)$ and $C_v=\left(\frac{k_v}{m_V\gamma_w}\right)$ are the consolidation coefficients in the radial and vertical coordinate directions, respectively.

2.1. Initial and Boundary Conditions

Though various loading models exist in the literature for different practical situations, a single ramp loading is adopted in this presentation. This model considers that the loading, $q(t)$, starts from zero level initially, and increases linearly with respect to time during the construction stage to its ultimate value, q_0, at some time t_0 (known as the construction duration) and remains constant beyond the construction duration. This is contrary to the uniformly distributed $q(t)$ proposed by Geng & Yu [25]. Thus, $q(t)$ is expressed as follows:

Initial Conditions

$$q(t)=\frac{q_0 t}{t_0},\ \text{for } 0\le t\le t_0;\ \text{and } q(t)=q_0,\ \text{for } t_0\le t\le\infty \tag{7}$$

The ramp loading surcharge given in Equation 7 is graphically depicted in Figure 3. Hence;

$$\frac{\partial q}{\partial t}=\frac{q_0}{t_0},\ \text{for } 0\le t\le t_0;\ \text{and } \frac{\partial q}{\partial t}=0 \text{ for } t>0 \tag{8}$$

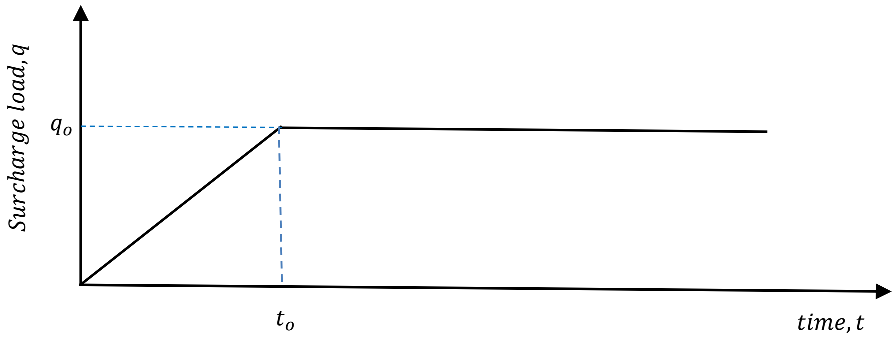

Figure 3. Ramp loading surcharge

By the implication of Equations 8, Equation 6 can be solved in two phases, namely the loading phase and the post-loading phase.

Boundary Conditions

In both cases, the boundary conditions are as follows.

- Due to the fact that the top is pervious and the bottom is impervious,

$$u(r,0,t)=\frac{\partial u(r,H,t)}{\partial z}=0 \tag{9}$$

- As a result of the excess pore water pressure at the drain well surface being zero,

$$u(a;\ z;\ t)=0 \tag{10}$$

- Due to symmetry, no flow occurs across the external boundary of the well. Hence,

$$\frac{\partial u(b,z,t)}{\partial z}=0 \tag{11}$$

In addition, for the loading phase problem, the initial excess pore water pressure is zero, that is,

$$u(r;\ z;\ 0)=0 \tag{12}$$

3. Problem Analysis and Solution

3.1. The Loading Phase Problem

The loading phase problem consists of the partial differential Equation 6 with the first of Equations 8, the boundary conditions given by Equations 9 to 11, and the initial condition expressed in Equation 12. The loading phase problem can be solved as follows:

First, the Laplace transformation of both sides of Equation 6 can be carried out to obtain the following equation:

$$C_r\left(\frac{\partial^2\bar{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{u}}{\partial r}\right)+C_v\frac{\partial^2\bar{u}}{\partial z^2}+\frac{Q}{s}\ =s\bar{u} \tag{13}$$

In Equation 13 above, $Q={q_o}/{t_o}$; s is the Laplace parameter and $\bar{u}=\int_0^\infty e^{-st}\,u(r;\ z;\ t)\,dt$ is the Laplace image of $u(r,z,t)$.

Second, carrying out the Finite Hankel Transformation of both sides of Equation 13 yields the following:

$$C_v\frac{\partial^2\tilde{\bar{u}}}{\partial z^2}-C_r\xi_i^2\tilde{\bar{u}}+\frac{\tilde{Q}}{s}\ =s\tilde{\bar{u}} \tag{14}$$

where, $\tilde{\bar{u}}=\int_a^b r\left(\frac{\partial^2\bar{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{u}}{\partial r}\right)\{J_o(\xi_i r)Y_o(\xi_i a)-J_o(\xi_i a)Y_o(\xi_i r)\}dr$ is the Finite Hankel transform of $\bar{u}(r,z,s)$. In addition, the quantities ξ_i are generated as the positive roots of the following eigenvalue problem in accordance with Cinelli [26]:

$$J_o(\xi_i a)Y_o'(\xi_i b)-Y_o(\xi_i a)J_o'(\xi_i b)=0 \tag{15}$$

Upon noting the relationships $J_o'(\xi_i x)=-J_1(\xi_i x)$ and $Y_o'(\xi_i x)=-Y_1(\xi_i x)$, Equation 15 is transformed to:

$$J_o(\xi_i a)Y_1(\xi_i b)-Y_o(\xi_i a)J_1(\xi_i b)=0 \tag{16}$$

In Equations 14 to 16, $J_o(\mu r)$ and $J_1(\mu r)$ are the Bessel functions of the first kind, of the orders zero and one, respectively. In the same vein, $Y_o(\mu r)$ and $Y_1(\mu r)$ are the Bessel functions of the second kind, of orders zero and one, respectively.

For further developments, the following notations are introduced:

$$\beta^2=\frac{1}{C_v};\text{ and }\alpha^2=s+C_r\xi_i^2 \tag{17}$$

By substituting appropriately from Equation 17 into Equation 14, with some re-arrangements, it yields:

$$\frac{\partial^2\tilde{\bar{u}}}{\partial z^2}-\beta^2\alpha^2\tilde{\bar{u}}=-\frac{\beta^2\tilde{Q}}{s} \tag{18}$$

Upon solving Equation 18, superimposing the complementary and particular solutions to obtain the general solution, we arrive at:

$$\tilde{\bar{u}}\ =\ A_1 cosh[\alpha\beta z]\ +\ A_2 sinh[\alpha\beta z]\ +\frac{\tilde{Q}}{\alpha^2 s} \tag{19}$$

where A_1 and A_2 are arbitrary constants to be determined. By a straightforward transformation of the boundary conditions into the Finite Hankel domain, A_1 and A_2 can be determined as follows:

$$A_1\ =\frac{\tilde{Q}}{\alpha^2 s}\text{ and }A_2=\frac{\tilde{Q}sinh[\alpha\beta z]}{\alpha^2 cosh[\alpha\beta z]} \tag{20}$$

In view of Equation 20, Equation 19 can be rewritten to read as follows

$$\tilde{\bar{u}}\ =\frac{\tilde{Q}sinh[\alpha\beta H]\,sinh[\alpha\beta z]}{s\alpha^2 cosh[\alpha\beta H]}-\frac{\tilde{Q}cosh[\alpha\beta z]}{s\alpha^2}+\frac{\tilde{Q}}{s\alpha^2} \tag{21}$$

which can be further simplified to read as follows:

$$\tilde{\bar{u}}\ =\frac{\tilde{Q}}{s\alpha^2}-\frac{\tilde{Q}cosh[\alpha\beta(H-z)]}{s\alpha^2 cosh[\alpha\beta H]} \tag{22}$$

Equation 22 gives an explicit expression for the image of $u(r;\ z;\ t)$ following the successive transformation of Laplace and Finite Hankel. Thus, by carrying out inversions of the transformation to obtain $u(r,z,t)$.

Let us first carry out the inversion of the Laplace transforms on both sides of Equation 22 to obtain:

$$\tilde{u} = \tilde{Q}\mathcal{L}^{-1}\left\{\frac{1}{s+C_r\xi_i^2}\right\} - \tilde{Q}\mathcal{L}^{-1}\left\{\frac{\cosh\left[\beta(H-z)\sqrt{s+C_r\xi_i^2}\right]}{s(s+C_r\xi_i^2)\cosh\left[\beta H\sqrt{s+C_r\xi_i^2}\right]}\right\} \tag{23}$$

where $\alpha^2$is replaced by $s + C_r\xi_i^2$ from Equation 17. Now, the first term on the right-hand side of Equation 23 equals $\tilde{Q}e^{-C_r\xi_i t}$. The second term is evaluated using the convolution theorem which states that:

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_o^t F(\tau)G(t-\tau)\,dt \tag{24}$$

In this case, $g(s) = \frac{1}{s}$, while

$$f(s) = \frac{\cosh\left[\beta(H-z)\sqrt{s+C_r\xi_i^2}\right]}{(s+C_r\xi_i^2)\cosh\left(\beta H\sqrt{s+C_r\xi_i^2}\right)} \tag{25}$$

The Laplace inverses of $f(s)$ and $g(s)$ are given respectively, as follows [27]:

$$F(t) = e^{-C_r\xi_i^2 t}\left\{1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}e^{-\left[\frac{(2n-1)^2\pi^2 t}{4\beta^2}\right]}\cos\frac{(2n-1)\pi(H-z)}{2H}\right\} \text{ and } G(t) = 1.0 \tag{26}$$

Hence,

$$\mathcal{L}\{f(s)g(s)\} = \int_0^t e^{-C_r\xi_i^2\tau}\left\{1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}e^{-\left[\frac{(2n-1)^2\pi^2 t}{4\beta^2}\right]}\cos\frac{(2n-1)\pi(H-z)}{2H}\right\}d\tau \tag{27}$$

Thus, evaluating the integral on the right-hand side of Equation 27, followed by some simplifications, and replacement of β^2 by $\frac{1}{C_v}$ we arrive at:

$$\tilde{u} = \frac{4\tilde{Q}}{\pi}\sum_{n=1}^{\infty}\frac{\left\{1-e^{\left[-C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right]t}\right\}\sin\left(n-\frac{1}{2}\right)\frac{\pi z}{H}}{(2n-1)\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\}} \tag{28}$$

By inverting the Finite Hankel transforms on both sides of Equation 28, following the procedure by Cinelli [26] to yield:

$$\begin{aligned} u(r,z,t) = 2\pi Q\sum_n\sum_{\xi_i}&\frac{\xi_i^2 J_1^2(\xi_i b)\int_a^b \varphi\{J_0(\xi_i\varphi)Y_0(\xi_i a)-J_0(\xi_i a)Y_0(\xi_i\varphi)\}d\varphi\sin\left(n-\frac{1}{2}\right)\frac{\pi z}{H}}{(2n-1)\{J_0^2(\xi_i a)-J_1^2(\xi_i b)\}\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\}}\times \\ &\times\left\{1-\exp-\left[C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right]t\right\}\{J_0(\xi_i r)Y_0(\xi_i a)-J_0(\xi_i a)Y_0(\xi_i r)\} \end{aligned} \tag{29}$$

Evaluating the integral on the right-hand side of Equation 29 yields:

$$\int_a^b \varphi\{J_0(\xi_i\varphi)Y_0(\xi_i a)-J_0(\xi_i a)Y_0(\xi_i\varphi)\} == \frac{1}{\xi_i}\{b[Y_0(\xi_i a)J_1(\xi_i b)-J_0(\xi_i a)Y_1(\xi_i b)]+a[J_0(\xi_i a)Y_1(\xi_i a)-Y_0(\xi_i a)J_1(\xi_i a)]\} \tag{30}$$

By virtue of Equation 16, the quantities in the first square brackets on the right-hand side of Equation 30 vanish leaving us with:

$$\int_a^b \varphi\{J_0(\xi_i\varphi)Y_0(\xi_i a)-J_0(\xi_i a)Y_0(\xi_i\varphi)\} = \frac{a}{\xi_i}[J_0(\xi_i a)Y_1(\xi_i a)-Y_0(\xi_i a)J_1(\xi_i a)] \tag{31}$$

Now, by substituting from Equation 31 into Equation 29 to arrive at Equation 32 which gives the final expression for the space-time variation of the pore water pressure in the loading stage of the consolidation process, that is, for the time interval, $0 \le t \le t_0$.

$$\begin{aligned} u(r,z,t) = 2\pi aQ\sum_{n=1}^{\infty}\sum_{\xi_i}&\frac{\xi_i J_1^2(\xi_i b)[J_0(\xi_i a)Y_1(\xi_i a)-Y_0(\xi_i a)J_1(\xi_i a)]\sin\left(n-\frac{1}{2}\right)\frac{\pi z}{H}}{(2n-1)\{J_0^2(\xi_i a)-J_1^2(\xi_i b)\}\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\}}\times \\ &\times\left\{1-\exp-\left[C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right]t\right\}\{J_0(\xi_i r)Y_0(\xi_i a)-J_0(\xi_i a)Y_0(\xi_i r)\} \end{aligned} \tag{32}$$

3.2. Post-Loading Phase Problem

This is the phase in which the load has reached the ultimate value with no further changes in the load, and the phase commences from time $t = t_0$. Consequently, for this phase, the governing differential Equation 6 reduces to the following:

$$C_r\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+C_v\frac{\partial^2 u}{\partial z^2}-\frac{\partial u}{\partial t}=0 \tag{33}$$

In Equation 33, $t_0 \le t < \infty$. In addition, the boundary conditions given by Equations 9 to 11 still hold for this phase. However, the initial condition for this phase can be obtained from the value of the pore pressure at $t = t_0$, as obtained from Equation 32. It follows then that the initial pressure for this phase is:

$$u_{t_0} = \quad u(r,z,t_0) = 2\pi a Q \sum_{n=1}^{\infty} \sum_{\xi_i}^{\infty} \frac{\xi_i J_1^2(\xi_i b)[J_0(\xi_i a)Y_1(\xi_i a)-Y_0(\xi_i a)J_1(\xi_i a)]\sin\left(n-\frac{1}{2}\right)\frac{\pi z}{H}}{(2n-1)\{J_0^2(\xi_i a)-J_1^2(\xi_i b)\}\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\}} \times$$
$$\times\left\{1-\exp-\left[C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right]t_0\right\}\{J_0(\xi_i r)Y_0(\xi_i a)-J_0(\xi_i a)Y_0(\xi_i r)\} \tag{34}$$

In order to solve Equation 33, it proves useful to transform time from the absolute scale into relative time variable defined as follows:

$$t' = t - t_0 \tag{35}$$

Therefore, Equation 33 transforms into the following:

$$C_r\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+C_v\frac{\partial^2 u}{\partial z^2}-\frac{\partial u}{\partial t'}=0 \tag{36}$$

Application of the Laplace transformation on Equation 36 yields the following equation.

$$C_v\frac{\partial^2 \mathrm{u}}{\partial z^2}+C_r\left(\frac{\partial^2 \mathrm{u}}{\partial r^2}+\frac{1}{r}\frac{\partial \mathrm{u}}{\partial r}\right)=s\mathrm{u}-u(t_0) \tag{37}$$

where $u(r,z,s)$ is the Laplace image of $u(r,z,t')$, and the quantity U_{t_0} is the value of $u(r,z,t')$ at $t'=0$. It should be noted that u_{t_0} can be obtained from either Equation 29 or 32 by setting $t = t_0$. Upon application of the Finite Hankel transformation to both sides of Equation 37, the following equation ensues:

$$C_v\frac{d^2\tilde{u}}{dz^2}-C_r\xi_i^2\tilde{u}=s\tilde{\mathrm{u}}-\tilde{u}(t_0) \tag{38}$$

In Equation 38, $\tilde{\mathrm{u}}(\xi_i,z,s)$ is the finite Hankel transform of $\mathrm{u}(r,z,s)$. By a straightforward reversal of the process that transformed Equation 27 to Equation 29, it is easily deduced that:

$$\tilde{u}(t_0)=u(r,z,t_0)=\frac{4\tilde{Q}}{\pi}\sum_{n=1}^{\infty}\frac{\left\{1-e^{-\left[C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right]t_0}\right\}\sin\frac{(2n-1)\pi z}{2H}}{(2n-1)\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\}} \tag{39}$$

Now, to re-write Equation 38 in terms of the notations introduced earlier in Equation 17 with some re-arrangements to read as follows:

$$\frac{d^2\tilde{\mathrm{u}}}{dz^2}-\beta^2\alpha^2\tilde{\tilde{u}}=-\beta^2\tilde{u}(t_0) \tag{40}$$

The second order ordinary differential Equation 40 admits general solution in the form,

$$\mathrm{u}(r,z,s)=B_1\cosh\beta\alpha z+B_2\sinh\beta\alpha z+B_3 \tag{41}$$

where B_1 and B_2 are arbitrary constants of integration from the complementary solution, and B_3 is the particular integral, given by:

$$B_3=\frac{4\tilde{Q}}{\pi}\sum_{n=1}^{\infty}\frac{\left\{1-e^{-\left[C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right]t_0}\right\}\sin\frac{(2n-1)\pi z}{2H}}{(2n-1)\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\}\left\{(s+C_r\xi_i^2)+\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\}} \tag{42}$$

Imposing the transformed boundary conditions:

$$\tilde{u}(\xi_i,0,s)=\frac{\partial\tilde{u}(\xi_i,H,s)}{\partial z}=0 \tag{43}$$

leads to $B_1 = B_2 = 0$. It therefore follows that

$$\tilde{\tilde{u}}(\xi_i, z, s) = \frac{4\tilde{Q}}{\pi}\sum_{n=1}^{\infty} \frac{\left\{1-e^{-\left[C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right]t_0}\right\}\sin\frac{(2n-1)\pi z}{2H}}{(2n-1)\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right\}\left\{(s+C_r\xi_i^2)+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right\}} \tag{44}$$

Inversion of the Laplace transforms on both sides of Equation 44, first, yields:

$$\tilde{u}(\xi_i, z, t') = \frac{4\tilde{Q}}{\pi}\sum_{n=1}^{\infty} \frac{e^{-\left[C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right]t'}\left\{1-e^{-\left[C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right]t_0}\right\}\sin\frac{(2n-1)\pi z}{2H}}{(2n-1)\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right\}} \tag{45}$$

Reverting to the absolute time domain by putting $t - t_0$ in place of t' in Equation 45 and simplifying the ensuing expression, we obtain:

$$\tilde{u}(\xi_i, z, t) = \frac{4\tilde{Q}}{\pi}\sum_{n=1}^{\infty} \frac{e^{-\left[C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right]t}\left\{e^{\left[C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right]t_0}-1\right\}\sin\frac{(2n-1)\pi z}{2H}}{(2n-1)\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right\}} \tag{46}$$

Finally, by carrying out inversion of the Finite Hankel Transformation on Equation 46 to obtain:

$$\begin{aligned} u(r,z,t) = {} & 2\pi a Q\sum_{n=1}^{\infty}\sum_{\xi_i}^{\infty}\left[(2n-1)\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right\}\{J_0^2(\xi_i a)-J_1^2(\xi_i b)\}\right]^{-1}\times \\ & \times \xi_i^2 J_1^2(\xi_i b)\times\exp-\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right\}t\times\left[\exp\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right\}t_0-1\right]\times \\ & \times\int_a^b p\{J_0(\xi_i p)Y_0(\xi_i a)-J_0(\xi_i a)Y_0(\xi_i p)\}dp\times[\{J_0(\xi_i r)Y_0(\xi_i a)-J_0(\xi_i a)Y_0(\xi_i r)\}]\sin\frac{(2n-1)\pi z}{2H} \end{aligned} \tag{47}$$

Evaluating the definite integral on the right-hand side of Equation 47 followed by some simplifications, in line with the procedure that led to Equation 31, yields

$$\begin{aligned} u(r,z,t) = {} & 2\pi a Q\sum_{n=1}^{\infty}\sum_{\xi_i}^{\infty}\left[(2n-1)\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right\}\{J_0^2(\xi_i a)-J_1^2(\xi_i b)\}\right]^{-1}\times \\ & \xi_i J_1^2(\xi_i b)\times\exp-\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right\}t\times\left[\exp\left\{C_r\xi_i^2+\frac{(2n-1)^2\pi^2C_v}{4H^2}\right\}t_0-1\right]\times \\ & [J_0(\xi_i a)Y_1(\xi_i a)-Y_0(\xi_i a)J_1(\xi_i a)]\times[\{J_0(\xi_i r)Y_0(\xi_i a)-J_0(\xi_i a)Y_0(\xi_i r)\}]\sin\frac{(2n-1)\pi z}{2H} \end{aligned} \tag{48}$$

Equation 48 gives the space-time variation of the pore pressure for the post-loading phase of the consolidation process. In the alternative, putting $N = (2n-1)$ Equation 48 takes the following form:

$$\begin{aligned} u(r,z,t) = {} & 2\pi a\frac{q_0}{t_0}\sum_{N=1,3,\dots}^{\infty}\sum_{\xi_i}^{\infty}\left[N\left\{C_r\xi_i^2+\frac{N^2\pi^2C_v}{4H^2}\right\}\{J_0^2(\xi_i a)-J_1^2(\xi_i b)\}\right]^{-1}\times \\ & \times\xi_i J_1^2(\xi_i b)\times\exp-\left\{C_r\xi_i^2+\frac{N^2\pi^2C_v}{4H^2}\right\}t\times\left[\exp\left\{C_r\xi_i^2+\frac{N^2\pi^2C_v}{4H^2}\right\}t_0-1\right]\times \\ & \times[J_0(\xi_i a)Y_1(\xi_i a)-Y_0(\xi_i a)J_1(\xi_i a)]\times[\{J_0(\xi_i r)Y_0(\xi_i a)-J_0(\xi_i a)Y_0(\xi_i r)\}]\sin\frac{N\pi z}{2H} \end{aligned} \tag{49}$$

4. Average Degree of Consolidation under Time- Dependent Surcharge

Defining the overall average excess pressure, in the whole of the undisturbed soil mass, as follows:

$$u_{ave} = \frac{1}{H}\int_0^H\left\{\frac{1}{\pi(b^2-a^2)}\int_a^b 2\pi r u(r,z,t)dt\right\}dz = \frac{2}{H^2(b^2-a^2)}\int_0^H\int_a^b r u(r,z,t)drdz \tag{50}$$

Then, the following defines the overall average degree of consolidation, for the single ramp loading surcharge [13, 16].

$$U(t) = \frac{1}{u_0}\left\{\frac{q(t)}{q_0}u_0 - u_{\text{ave}}(t)\right\}, 0 \le t \le t_0;\ \text{for } t \le t_0\ u(t) = 1 - \frac{u_{\text{ave}}}{u_0}\ \text{for } t \ge t_0 \tag{51}$$

where u_0 is the ultimate pore pressure, which in this case equals q_0. In view of Equations 6 and 50, Equation 51 is rewritten as follows:

$$U(t) = \begin{cases} \frac{t}{t_0} - \frac{2}{H^2(b^2-a^2)}\int_0^H\int_a^b r u(r,z,t)drdz,\ 0 \le t \le t_0 \\ 1 - \frac{2}{H^2(b^2-a^2)}\int_0^H\int_a^b r u(r,z,t)drdz\ t \ge t_0 \end{cases} \tag{52}$$

By substituting for $u(r,z,t)$ from Equations 32 and 48 into the first and second of Equations 52, respectively to obtain:

$$U(T)=\begin{cases}\frac{T}{T_0}-\frac{8}{aT_0(K^2-1)}\sum_{n=1}^{\infty}\sum_{\mu_i}\frac{\mu_i J_1^2(K\mu_i)\{J_0(\mu_i)Y_1(\mu_i)-J_1(\mu_i)Y_0(\mu_i)\}^2(1-e^{-\Omega_{in}T})}{(2n-1)^2\Omega_{in}\{J_0^2(\mu_i)-J_1^2(K\mu_i)\}}, 0\le T\le T_0\\ 1-\frac{8}{aT_0(K^2-1)}\sum_{n=1}^{\infty}\sum_{\mu_i}\frac{\mu_i J_1^2(K\mu_i)\{J_0(\mu_i)Y_1(\mu_i)-J_1(\mu_i)Y_0(\mu_i)\}^2(1-e^{-S_{in}T})}{(2n-1)^2\Omega_{in}\{J_0^2(\mu_i)-J_1^2(K\mu_i)\}}, T\ge T_0\end{cases} \quad (53)$$

The new symbols appearing in Equations 53, are defined as follows:

$$T=\frac{\pi^2}{4H^2}C_v t; T_0=\frac{\pi^2}{4H^2}C_v t_0;\ \mu_i=\xi_i a;\ K=\frac{b}{a};\ \Omega_{in}=\frac{4C_r H^2}{\pi^2 C_v a^2}\mu_i^2+(2n-1)^2 \quad (54)$$

In the alternative, Equations 53 can be expressed as follows:

$$U(T)=\begin{cases}\frac{T}{T_0}-\frac{8}{aT_0(K^2-1)}\sum_{N=1,3,5,\dots}^{\infty}\sum_{\mu_i}\frac{\mu_i J_1^2(K\mu_i)\{J_0(\mu_i)Y_1(\mu_i)-J_1(\mu_i)Y_0(\mu_i)\}^2(1-e^{-\Omega_{in}T})}{N^2\Omega_{in}\{J_0^2(\mu_i)-J_1^2(K\mu_i)\}}, 0\le T\le T_0\\ 1-\frac{8}{aT_0(K^2-1)}\sum_{N=1,3,5,\dots}^{\infty}\sum_{\mu_i}\frac{\mu_i J_1^2(K\mu_i)\{J_0(\mu_i)Y_1(\mu_i)-J_1(\mu_i)Y_0(\mu_i)\}^2(1-e^{-\Omega_{in}T})}{N^2\Omega_{in}\{J_0^2(\mu_i)-J_1^2(K\mu_i)\}}, T\ge T_0\end{cases} \quad (55)$$

Now, for a numerical illustration of the solution, especially Equations 53, let us consider the case in which, $H=5.0$ m; $a=0.15$ m; $C_v=1.125$ m^2/ year $C_r=2.5$ m^2/ year; $t_0=1$ year. Hence $T=0.111t$, and $T_0=0.111$, where t is in years, and T is the dimensionless time factor. We shall examine three scenarios corresponding to $K\in\{10,15,20\}$. We note here that the eigenvalue Equation 16 transforms to the following, in terms of μ_i, as defined in Equation 54:

$$J_0(\mu_i)Y_1(K\mu_i)-Y_0(\mu_i)J_1(K\mu_i)=0 \quad (56)$$

The eigenvalues of Equation 56, corresponding to the different values of K, are shown in Table 1 below.

Table 1. The eigenvalues of Equation 56 with corresponding values of K

K	*μ1*	*μ2*	*μ3*	*μ4*	*μ5*	*μ6*	*μ7*
10	0.11027	0.49788	0.85543	1.20868	1.56029	1.91107	2.26138
15	0.06612	0.31680	0.54712	0.77457	1.00090	1.22663	1.45202
20	0.04651	0.23175	0.40160	0.56934	0.73622	0.90266	1.06883

In Figure 4, the values of $U(T)$ are plotted of as a function of the time factor T, for different values of K, the aspect ratio.

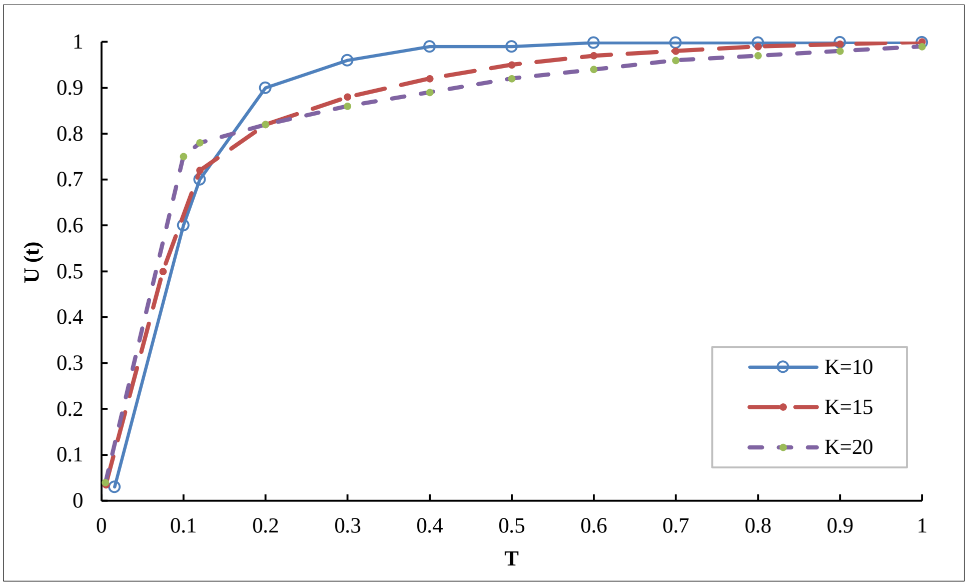

Figure 4. Average Degree of Consolidation vs. Time Factor

5. Validation Check for the Solution Procedure

5.1. *Case 1:* Vertical and Radial Drainage under Instantaneous Loading ($t_0\to 0$)

Firstly, the validation check for the solution procedure is approached by examining Equation 49 in the context of the limiting case of $t_0\to 0$. This scenario coincides with the case whereby the ultimate loading, q_0, is instantaneously applied at the commencement of the consolidation process, without any time variation thereafter.

Clearly, Equation 49 in explicit term of t_0 takes the form:

$$u(r,z,t,t_0) = \frac{g(t_0)}{h(t_0)} \tag{57}$$

where;

$$g(t_0) = 2\pi a q_0 \sum_{n=1}^{\infty} \sum_{\xi_i}^{\infty} \left[(2n-1)\left\{C_r\xi_i^2 + \frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\}\{J_0^2(\xi_i a) - J_1^2(\xi_i a)\}\right]^{-1} \times$$
$$\times \xi_i J_1^2(\xi_i b) \times \exp - \left\{C_r\xi_i^2 + \frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\} t \times \left[\exp\left\{C_r\xi_i^2 + \frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\} t_0 - 1\right] \times$$
$$\times [J_0(\xi_i a)Y_1(\xi_i a) - Y_0(\xi_i a)J_1(\xi_i a)] \times [\{J_0(\xi_i r)Y_0(\xi_i a) - J_0(\xi_i a)Y_0(\xi_i r)\}] \sin\frac{(2n-1)\pi z}{2H} \tag{58}$$

and,

$$h(t_0) = t_0 \tag{59}$$

Direct evaluation of the $\lim_{t_0\to 0} u(r,z,t,t_0)$ using Equation 57 leads to the special case of $\frac{0}{0}$, thereby yielding no useful information. Therefore, we apply the l'Hopital rule which gives the limiting value of u(r, z, t, t_0) given by Equation 57 through the following expression

$$\lim_{t_0\to 0} u(r,z,t,t_0) = \lim_{t_0\to 0} \frac{g'(t_0)}{h'(t_0)} \tag{60}$$

where primes indicate differentiation with respect to t_0. Thus,

$$\lim_{t_0\to 0} u(r,z,t,t_0) = 2\pi a q_0 \sum_{n=1}^{\infty} \sum_{\xi_i}^{\infty} [(2n-1)\{J_0^2(\xi_i a) - J_1^2(\xi_i b)\}]^{-1} \times \xi_i J_1^2(\xi_i b) \times \exp - \left\{C_r\xi_i^2 + \frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\} t$$
$$\times [J_0(\xi_i a)Y_1(\xi_i a) - Y_0(\xi_i a)J_1(\xi_i a)] \times [\{J_0(\xi_i r)Y_0(\xi_i a) - J_0(\xi_i a)Y_0(\xi_i r)\}] \sin\frac{(2n-1)\pi z}{2H} \tag{61}$$

Again, going further to define $N = 2n-1$, we can express Equation 61 in the following alternative form:

$$\lim_{t_0\to 0} u(r,z,t,t_0) = 2\pi a q_0 \sum_{N=1,3,5,\ldots}^{\infty} \sum_{\xi_i}^{\infty} [N\{J_0^2(\xi_i a) - J_1^2(\xi_i b)\}]^{-1} \times \xi_i J_1^2(\xi_i b) \times \exp - \left\{C_r\xi_i^2 + \frac{N^2\pi^2 C_v}{4H^2}\right\} t$$
$$\times [J_0(\xi_i a)Y_1(\xi_i a) - Y_0(\xi_i a)J_1(\xi_i a)] \times [\{J_0(\xi_i r)Y_0(\xi_i a) - J_0(\xi_i a)Y_0(\xi_i r)\}] \sin\frac{N\pi z}{2H} \tag{62}$$

Equation 61 or its alternative form 62 is indeed the solution of the consolidation problem due to step (instantaneous) loading.

5.2. *Case 2:* Vertical Consolidation for Instantaneous Loading Only

Secondly, we shall examine the procedure in the context of vertical consolidation only for instantaneous loading. In other words, the case where radial consolidation is negligible. To do this we shall go back to Equation 45 and set $C_r = 0, t_0 \to 0$, and hence $t' = t$. In this case, the finite Hankel transform does not apply. Thus, from 45 we generate the following:

$$u(z,t) = \lim_{t_0\to 0, C_r=0} \frac{4q_0}{\pi t_0} \sum_{n=1}^{\infty} \frac{e^{-\left[C_r\xi_i^2 + \frac{(2n-1)^2\pi^2 C_v}{4H^2}\right]t'} \left\{1 - e^{-\left[\left[C_r\xi_i^2 + \frac{(2n-1)^2\pi^2 C_v}{4H^2}\right]t_0\right.}\right\} \sin\frac{(2n-1)\pi z}{2H}}{(2n-1)\left\{C_r\xi_i^2 + \frac{(2n-1)^2\pi^2 C_v}{4H^2}\right\}} \tag{63}$$

Again, upon application of the l'Hopital rule to Equation 63 one obtains the following:

$$u(z,t) = \frac{4q_0}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\left[\frac{(2n-1)^2\pi^2 C_v}{4H^2}\right]t} \sin\frac{(2n-1)\pi z}{2H}}{(2n-1)} \tag{64}$$

In the alternative form, Equation 64 can be expressed as:

$$u(z,t) = \frac{4q_0}{\pi} \sum_{N=1,3,5\ldots}^{\infty} \frac{e^{-\left[\frac{N^2\pi^2 C_v}{4H^2}\right]t} \sin\frac{N\pi z}{2H}}{N} \tag{65}$$

Equation 65 expresses the pore pressure variation with respect to space and time for a 1-dimensional consolidation process (under instantaneous loading) as earlier developed by Terzaghi, et al. [28].

5.3. *Case 3:* Average Degree of Consolidation by Radial Drainage Only

Thirdly, we shall examine a numerical example earlier considered by Leo [16] for the case of average degree of consolidation by radial drainage only, associated with instantaneous loading using the following input data:

$C_r = \frac{7.9 \text{ m}^2}{\text{year},\ H} = 0$ m, $a = 0.2$ m, $b = 1.8$ m.

It follows then that $K = 9$. Besides, we shall take the radial consolidation time factor T_r to be equal to $\frac{C_r t}{4b^2}$ (in line with Leo [16].

By this development, the excess pore pressure u is given by;

$$u(r,t) = \frac{q_0 \pi^2 a}{2} \sum_{\xi_i} \frac{\xi_i J_1^2(\xi_i b)\{J_0(\xi_i a) Y_1(\xi_i a) - Y_0(\xi_i a) J_1(\xi_i a)\}}{J_0^2(\xi_i a) - J_1^2(\xi_i b)} \times \{J_0(\xi_i r) Y_0(\xi_i a) - J_0(\xi_i a) Y_0(\xi_i r)\} e^{-C_r \xi_i t} \tag{66}$$

The average degree of radial consolidation is then given by;

$$\begin{aligned} U_r(T_r) &= 1 - \frac{2\int_a^b u(r,T_r) dr}{(b^2-a^2) q_0} = 1 - \frac{\pi^2 a^2}{(b^2-a^2)} \sum_{\xi_i}^{\infty} \frac{J_1^2(\xi_i b)\{J_0(\xi_i a) Y_1(\xi_i a) - Y_0(\xi_i a) J_1(\xi_i a)\} e^{-C_r \xi_i t}}{J_0^2(\xi_i a) - J_1^2(\xi_i b)} \\ &= 1 - \frac{\pi^2}{(K^2-1)} \sum_{\xi_i}^{\infty} \frac{J_1^2(K\mu_i)\{J_0(\mu_i) Y_1(\mu_i) - Y_0(\mu_i) J_1(\mu_i)\} e^{-4T_r K^2 \mu_i^2}}{J_0^2(\mu_i) - J_1^2(K\mu_i)} \end{aligned} \tag{67}$$

The results obtained from the present theory and those of Leo [16] are depicted in the graph in Figure 5, which shows a good agreement between the two. In fact, whatever minor deviations there are between the two results, especially for low-time factors, can be attributed to the well resistance and smear effects considered by Leo [16] but ignored in this presentation. Since Leo [16] demonstrated the agreement between his results and those of Barron & Hansbo [5, 29], by implication the results of this presentation also agree with Barron & Hansbo [5, 29].

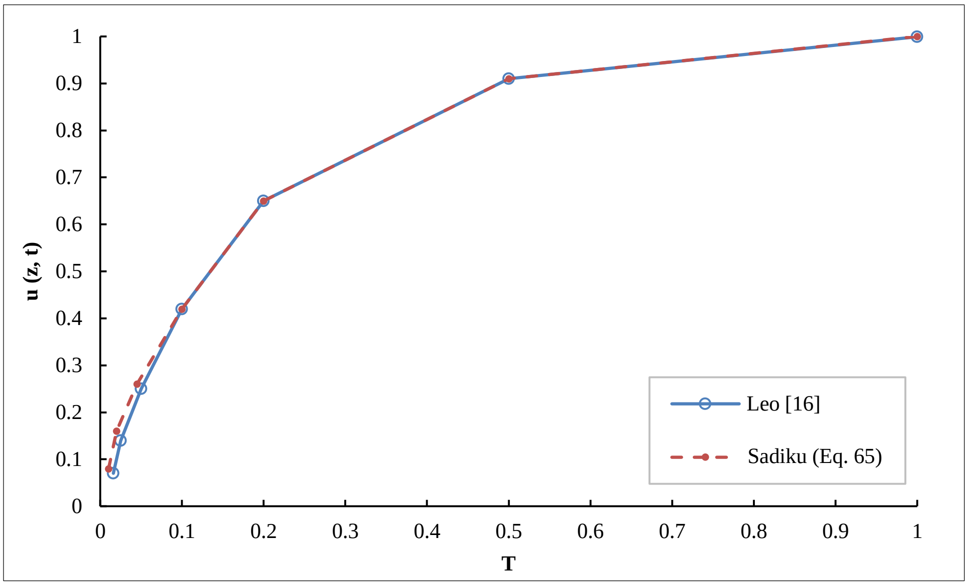

Figure 5. Comparison of Degree of Radial Consolidation

These three scenarios provide some form of verification for the validity of the theory being enunciated in this presentation. Moreover, numerical applications of this presentation prove to be straightforward and pose no significant difficulties.

6. Conclusion

A rigorous analysis of the problem of combined vertical and radial consolidation of soils under the action of a time-dependent surcharge has been presented. The theory advanced herein treats the radial and vertical flows as a single coupled process, driven by one pore water pressure. The concept of handling the bipartite flow without introducing any form of decoupling confers distinctiveness on this presentation. The solution methodology fundamentally rests upon successive application of the Laplace and finite Hankel transformations, leading to an explicit expression for the image of the pore water pressure as a function of the integral transforms' parameters. By subsequently inverting the integral transforms a closed-form solution in the sense of a generalized Fourier series is obtained for the pore water pressure. The ensuing solution offers a convenient route for obtaining the average degree of consolidation. Despite basing the analysis on the single ramp time-dependent surcharge, the method is applicable to any other type of time-dependent surcharge. The validity of the solution obtained has been established through logical checks and comparison with the results of previously reported research. In addition, a numerical illustration of the development has been made through the use of specific values of the associated parameters. In particular, the average degree of radial consolidation obtained from the present theory has been tested with Leo (2004) development, and the two results show very good agreement.

7. Notations

a, b	Inner and outer radii of a typical unit drainage cell	k_r, k_v	Permeability coefficients in the radial and vertical directions
C_r, C_v	Consolidation coefficients in the radial and vertical directions	r, z	Spatial coordinates
t	Temporal coordinate	t_0	Pre-loading duration
T	Dimensionless time factor corresponding to t	T_0	Dimensionless time factor corresponding to t_0
m_V	Compressibility coefficient	$q(t), q_0$	Rate of loading; value of q, at t_0
γ_w	Unit weight of water	u	Pore water pressure
$u, \tilde{u}$	Laplace and finite Hankel images of u	s	Laplace transform parameter
J_v, Y_v	Bessel functions of order v, and of the First and Second Kind	J'_v, Y'_v	Derivatives of Bessel functions of the First and Second Kind
ξ_i	Roots of the finite Hankel eigenvalue equation	μ_i	Factored ξ_i
α, β	Factored C_r and C_v	K	Aspect ratio, that is $\frac{b}{a}$
H	Depth of the consolidating soil	L, L^{-1}	Laplace transform and inverse Laplace transform
Q	$\frac{q_0}{t_0}$	Ω_{in}	Factored $\frac{C_r}{C_v}$
ε	Strain	N	$2n-1$

8. Declarations

8.1. Data Availability Statement

The data presented in this study are available in the article.

8.2. Funding

The author received no financial support for the research, authorship, and/or publication of this article.

8.3. Conflicts of Interest

The author declares no conflict of interest.

9. References

[1] Meilani, A., Ahmad, R., & Fikri, F. (2020). Analysis of settlement prediction due to preloading and vertical drain applications on runway construction. E3S Web of Conferences, 156, 2002. doi:10.1051/e3sconf/202015602002.

[2] Kjellman, W. (1948). Accelerating consolidation of fine grain soils by means of cardboard wicks. Proceedings of the Second International Conference on Soil Mechanics and foundation Engineering, 21-30June, 1948, Rotterdam, Netherlands.

[3] Terzaghi, K. (1943). Theoretical Soil Mechanics. John Wiley & Sons, Hoboken, United States. doi:10.1002/9780470172766.

[4] Horne, M. R. (1964). The consolidation of a stratified soil with vertical and horizontal drainage. International Journal of Mechanical Sciences, 6(2), 187–197. doi:10.1016/0020-7403(64)90015-3.

[5] Barron, R. A. (1948). Consolidation of Fine-Grained Soils by Drain Wells by Drain Wells. Transactions of the American Society of Civil Engineers, 113(1), 718–742. doi:10.1061/taceat.0006098.

[6] Walker, R., Indraratna, B., & Sivakugan, N. (2009). Vertical and Radial Consolidation Analysis of Multilayered Soil Using the Spectral Method. Journal of Geotechnical and Geoenvironmental Engineering, 135(5), 657–663. doi:10.1061/(asce)gt.1943-5606.0000075.

[7] Carrillo, N. (1942). Simple Two and Three Dimensional Case in the Theory of Consolidation of Soils. Journal of Mathematics and Physics, 21(1–4), 1–5. doi:10.1002/sapm19422111.

[8] Olson, R. E. (1977). Consolidation under Time Dependent Loading. Journal of the Geotechnical Engineering Division, 103(1), 55–60. doi:10.1061/ajgeb6.0000369.

[9] Conte, E., & Troncone, A. (2009). Radial consolidation with vertical drains and general time-dependent loading. Canadian Geotechnical Journal, 46(1), 25–36. doi:10.1139/T08-101.

[10] Chai, J. C., Shen, S. L., Miura, N., & Bergado, D. T. (2001). Simple method of modeling PVD-improved subsoil. Journal of geotechnical and geoenvironmental engineering, 127(11), 965-972. doi:10.1061/(ASCE)1090-0241(2001)127:11(965).

[11] Lee, C., Choi, Y., Lee, W., & Hong, S. J. (2013). Evaluation of vertical drain-enhanced radial consolidation with modified analytical solution. 18th International Conference on Soil Mechanics and Geotechnical Engineering: Challenges and Innovations in Geotechnics, ICSMGE 2013, 2-6 September, 2013, Paris, France.

[12] Hsu, T.-W., & Tsai, T.-H. (2016). Combined Vertical and Radial Consolidation under Time-Dependent Loading. International Journal of Geomechanics, 16(3). doi:10.1061/(asce)gm.1943-5622.0000600.

[13] Tang, X. W., & Onitsuka, K. (2000). Consolidation by vertical drains under time-dependent loading. International Journal for Numerical and Analytical Methods in Geomechanics, 24(9), 739-751. doi:10.1002/1096-9853(20000810)24:9<739::AID-NAG94>3.0.CO;2-B.

[14] Chi, Y., Tam, N. F. Y., Li, W. C., & Ye, Z. (2022). Multiple geochemical and microbial processes regulated by redox and organic matter control the vertical heterogeneity of As and Cd in paddy soil. Science of the Total Environment, 156229. doi:10.1016/j.scitotenv.2022.156229.

[15] Zhu, G., & Yin, J. H. (2004). Accuracy of Carrillo's formula for consolidation of soil with vertical and horizontal drainage under time-dependent loading. Communications in Numerical Methods in Engineering, 20(9), 721–735. doi:10.1002/cnm.706.

[16] Leo, C. J. (2004). Equal Strain Consolidation by Vertical Drains. Journal of Geotechnical and Geoenvironmental Engineering, 130(3), 316–327. doi:10.1061/(asce)1090-0241(2004)130:3(316).

[17] Sadiku, S. (2013). Analytical and computational procedure for solving the consolidation problem of layered soils. International Journal for Numerical and Analytical Methods in Geomechanics, 37(16), 2789–2800. doi:10.1002/nag.2162.

[18] Zhu, G., & Yin, J. H. (1998). Consolidation of soil under depth-dependent ramp load. Canadian Geotechnical Journal, 35(2), 344–350. doi:10.1139/t97-092.

[19] Lei, G. H., Zheng, Q., Ng, C. W. W., Chiu, A. C. F., & Xu, B. (2015). An analytical solution for consolidation with vertical drains under multi-ramp loading. Geotechnique, 65(7), 531–541. doi:10.1680/geot.13.P.196.

[20] AKAN, R., & SERT, S. (2021). Investigation of the Consolidation Behavior of Soft Soil Improved with Vertical Drains by Finite Element Method. International Journal of Engineering and Applied Sciences, 13(3), 93–105. doi:10.24107/ijeas.1002115.

[21] Turukmane, R., Gulhane, S., Kolte, P., & Chaudhary, B. (2019). Prefabrication vertical drain for soft soil consolidation. Technical Textiles International, 62, 27–29.

[22] Chen, L., Gao, Y., Elsayed, A., & Yang, X. (2019). Soil Consolidation and Vacuum Pressure Distribution under Prefabricated Vertical Drains. Geotechnical and Geological Engineering, 37(4), 3037–3048. doi:10.1007/s10706-019-00822-3.

[23] García-Ros, G., Alhama, I., & Cánovas, M. (2018). Powerful software to Simulate soil Consolidation problems with prefabricated Vertical Drains. Water (Switzerland), 10(3), 242. doi:10.3390/w10030242.

[24] Indraratna, B. (2017). Recent advances in vertical drains and vacuum preloading for soft ground stabilisation. Proceedings of 19th International Conference on Soil Mechanics and Geotechnical Engineering, 17-22 September, 2017, Seoul, South Korea.

[25] Geng, X., & Yu, H. S. (2017). A large-strain radial consolidation theory for soft clays improved by vertical drains. Geotechnique, 67(11), 1020–1028. doi:10.1680/jgeot.15.T.013.

[26] Cinelli, G. (1965). An extension of the finite Hankel transforms and applications. International Journal of Engineering Science, 3(5), 539–559. doi:10.1016/0020-7225(65)90034-0.

[27] Spiegel, M. R., Lipschutz, S., & Liu, J. (2009). Schaum's Outlines: Mathematical Handbook of Formulas and Tables. McGraw-Hill, New York City, United States.

[28] Najdanovic, N., & Obradovic, R. (1996). Soil mechanics in engineering practice. John Wiley & Sons, Hoboken, United States. doi:10.1097/00010694-194911000-00029.

[29] Hansbo, S. (1981). Consolidation of fine-grained soils by prefabricated drains. Proceedings of the 10th International Conference on Soil Mechanics and Foundation Engineering, 15-19 June, 1981, Stockholm, Sweden.